1. What is a standardised score and why is this helpful?
2. What is a z-score?
3. How do we transform an original score into a z-score?
4. What is the rough interpretation of a z-score? What does a z-score of 0 stand for?
5. How do we transform z-scores into raw scores?
6. What is the normal distribution?
7. What is the probability of an outcome?
8. How do we calculate the probability?
9. What is the standard normal distribution and why do we use it?
10. How do we calculate the probability associated with a z-score?
11. What is a percentile rank and what is a percentile?
5.1 The need for standardised scores

In the previous chapters we have seen that scientists have a pretty accurate picture of a sample when they know the central tendency (typically measured with the mean) and the variability (typically measured with the standard deviation). Therefore, no serious psychologist will be satisfied with a statement like: ‘The child had a score of 57 on the sit-up test.’ The first questions would be ‘What was the mean of the test and what was the standard deviation?’ Only with that knowledge can the psychologist know whether the child performed well.

Because scientists always try to communicate their findings as succinctly as possible, they have looked for a single measure that indicates the position of an individual within a distribution, one that does not depend on the mean and the standard deviation of the specific distribution. This score is known as the standardised score, or the \( z \)-score. It allows researchers to immediately assess the position of an individual within a distribution because it has the same (standardised) meaning for all distributions. So, whereas you cannot usefully report that a child has a score of 57 on a sit-up test, you can tell a psychologist that the child has a \( z \)-score of +1.65. This will be enough for them to understand that the child did very well. (As a matter of fact, only about 5% of children are expected to do better.)

In addition, by using standardised scores, psychologists can immediately compare a child’s performance on different types of test. So, when they are told that a child has a \( z \)-score of +1.65 on a sit-up test and a \( z \)-score of –0.5 on a reasoning test, they immediately know that the child did much better on the sit-up test than on the reasoning test. In this chapter, we will see how \( z \)-scores are calculated and interpreted.

5.2 Transforming raw scores into \( z \)-scores

The position of an individual within a group is defined in terms of the mean and the standard deviation of the group. We can use these two measures to modify the individual’s original score so that all scores have the same, standardised, meaning, which is clear to everyone who has taken a course in statistics. To do so, we calculate the \( z \)-score.

The \( z \)-score is a transformation of the original score so that the mean is zero and the standard deviation is 1. It is calculated from:

\[
z = \frac{X - M}{SD}
\]

in which \( X \) is the original score, \( M \) is the mean, and \( SD \) is the standard deviation.

For example, suppose Ms Brewer obtained a score of 26 on her psychology exam, and Ms Davidson obtained a score of 50. Knowing that the mean score

---

\[1\] This only applies to data from an interval/ratio measurement scale. Notice that in the remainder of this chapter, we assume that the data are from such a scale.
for the exam was 44 and the standard deviation was 20, we can calculate the $z$-scores as follows:

for Ms Brewer:  
$$z = \frac{26 - 44}{20} = \frac{-18}{20} = -0.9$$

for Ms Davidson:  
$$z = \frac{50 - 44}{20} = \frac{6}{20} = +0.3$$

### 5.3 Interpreting $z$-scores

The interpretation of a $z$-score involves two parts. First, the sign of the $z$-score tells us whether the score is located above (+) or below (–) the mean. A positive $z$-score means that the individual did better than average; a negative $z$-score means that the individual did worse. A $z$-score of 0 means that the individual's performance was right on the mean of the population. Applied to our example of the psychology exam, Ms Brewer did worse than average, whereas Ms Davidson did better.

The second part of the interpretation involves the numerical value of the $z$-score. This value tells us how far the score is from the mean in terms of the number of standard deviations. The bigger the value, the more extreme the score. As you may remember from the previous chapter, a $z$-score of –2 is pretty bad. It indicates an individual whose performance is two standard deviations below the mean. (This is among the worst performances you would expect in the sample.) Conversely, a $z$-score of +2 is very good, because few people are expected to do better than the mean plus twice the standard deviation. Applied to the example of the psychology exam, it is clear that the scores of both Ms Brewer and Ms Davidson were much less extreme. Whereas Ms Brewer was nearly one standard deviation below the mean ($z = -0.9$), Ms Davidson’s score was less than half a standard deviation above the mean ($z = +0.3$). As a rule of thumb, psychologists consider $z$-scores between –1.0 and +1.0 as ‘around the average’. They are fairly typical of what can be expected – not much better or worse than would be expected. This usually implies that no further action is required.

In summary, when we interpret a $z$-score, we first look at the sign, to see whether the individual belongs to the upper half of the group (+ score) or to the lower half (– score). Then we look at the magnitude of the score. As a rule of thumb, the following broad categories can be used:

- $z$-score lower than –2: very bad performance (much worse than expected)
- $z$-score between –2 and –1: quite bad performance (certainly when the $z$-score approaches –2)
Standardised scores, normal distribution and probability

- **z-score between –1 and +1:** typical performance (as expected)
- **z-score between +1 and +2:** quite good performance (the better as the score comes closer to +2)
- **z-score above +2:** very good performance

You can also use these categories to check whether your calculation of a z-score is correct. A negative z-score means that the performance of the individual must be worse than the mean performance; a positive z-value means that the individual's score is higher than the mean performance. The more extreme the z-score, the bigger the deviation from the mean (relative to the standard deviation). As a rule, when you obtain a z-score outside the region between –2 and +2, you should check it carefully because these scores should not occur very often. (You may find large z-scores if you are working with special groups; for instance, a group of children with special educational needs taking an intelligence test.)

Because all z-scores have the same meaning, they allow us to rapidly compare scores on different tests. For example, suppose Mr Camp scored 24 on a test of emotional intelligence (which has a mean of 30 and a SD of 5) and 12 on a test of academic intelligence (which has a mean of 10 and a SD of 1). Are the performances of Mr Camp comparable over the two tests? To answer this question, we calculate the z-scores:

- **Emotional intelligence:**  
  \[ z = \frac{24 - 30}{5} = \frac{-6}{5} = -1.2 \]
- **Academic intelligence:**  
  \[ z = \frac{12 - 10}{1} = \frac{2}{1} = +2.0 \]

On the basis of this information, we immediately see that Mr Camp scored very high on the test of academic intelligence (+2.0 is the start of the exceptional range), whereas he scored quite low on the test of emotional intelligence. Notice that Mr Camp’s raw scores went in the opposite direction: The raw score on the test of emotional intelligence (24) was higher than the raw score on the test of academic intelligence (12). However, a comparison of these raw scores makes little sense, as the means and the standard deviations of both tests vary widely. In contrast, the means and standard deviations of the z-scores are, by definition, the same and can be compared.

### 5.4 Transforming z-scores into raw scores

When we know the z-score of an individual, together with the mean (M) and the standard deviation (SD) of the distribution, we can calculate the individual’s raw X-score. For this, we will use the equation:

\[ X = M + z \times SD \]

For example, suppose Ms Innis had a z-score of –1.2 on a test with M = 30 and SD = 5. Then, we can calculate Ms Innis’s original, raw, score as follows:

\[ X = 30 + (-1.2 \times 5) = 30 + (-6) = 24 \]
As a calculation check, we use our knowledge that a negative z-score must result in a raw score below the mean and a positive z-score in a raw score above the mean. Given that Ms Innis’s z-score was negative (–1.2), her raw score (24) must be lower than the mean (30). In addition, her raw score must be more than 1 standard deviation below the mean, thus lower than $30 - 5 = 25$. Doing this type of elementary calculation checks will ensure that you never come up with solutions that are completely unacceptable.

Thus far, we have talked about z-scores in broad, intuitive terms as scores that refer to very bad results, acceptable results, or very good results. However, as we shall see, z-scores can provide us with much more detailed information about the position of an individual within the population if we can assume that the distribution of the scores is a normal distribution. Therefore, we now turn to a discussion of the normal distribution.

5.5 The normal distribution

5.5.1 Many variables form a normal distribution

In the 19th century, scientists discovered that many biological and behavioural characteristics follow a similar frequency distribution. For many phenomena, there are a lot of observations in a restricted range of values and outside this range the frequencies drop in a consistent and more or less symmetrical way. For instance, when we look at the heights of 50-year old men, we see that many of them are between 170 cm and 180 cm high (see Figure 5.1). The numbers of men smaller than 170 cm gradually decrease until there are virtually none.

Figure 5.1 Empirical frequency distribution of the height of American men in the 1970s, based on a sample of over 10,000 men. Notice that the frequencies are given in percentages rather than in raw numbers. This is typical for a relative frequency distribution. Also notice the theoretical, normal distribution, assumed to be the distribution of the heights of the population of all American adult men at that age.
Standardised scores, normal distribution and probability

smaller than 150 cm. Similarly, the numbers of men taller than 180 cm gradually decrease, and there are virtually none taller than 200 cm.2

A similar distribution is found when a group of students take an exam (see Figure 5.2). Only a few students do not manage to solve a single problem. Similarly, no students have all answers correct (at least not when the test is well constructed, so that it is not too easy). Instead, there is a mode at a certain performance level and the frequencies of results around this mode quite closely follow the bell-shaped distribution shown in Figure 5.1.

5.5.2 The shape of a normal distribution

Mathematicians discovered that this type of distribution will be observed each time a phenomenon is the result of several independent influences.3 The appropriate mathematical definition of the underlying, theoretical distribution is known as the normal distribution.

The normal distribution is a theoretical frequency distribution that has a bell shape, as shown in Figure 5.3. Notice that this theoretical frequency distribution no longer consists of the rectangular bars we have come to expect for empirical

2 The same need not be true for their sons as there has been an average increase of 3–4 cm per generation over recent decades.

3 For those who are interested, the contribution of several independent variables can be approximated by a situation in which a number of independent binary decisions are made (e.g., choose at random left or right). On the internet you find several website applets that simulate this aspect (see the website for this book – www.palgrave.com/psychology/brysbaert – for links). Notice how closely the end result resembles a normal distribution, certainly when the number of choices is large enough.
Standardised scores, normal distribution and probability

frequency distribution graphs (as in Figures 5.1 and 5.2). This is because we assume the bars to be so small that they are no longer ‘perceivable’. It is the distribution you would expect to find if you were able to examine the entire population from which the observed sample was drawn (see Chapter 1 for a discussion of the distinction between population and sample) and if you were able to measure the X-scores with very high (infinite) precision (see Chapter 2).

It is customary in statistics to use Greek symbols to refer to population measures. So, we will use the Greek letter μ (mu, pronounced ‘myoo’) to refer to the population mean, and the Greek letter σ (sigma, pronounced ‘myoo’) to refer to the population standard deviation.

When a variable is assumed to be normally distributed at the population level, we expect the variable to show the following characteristics:

- The frequency distribution of the values is symmetrical. The left side is a mirror image of the right side.
- The frequency distribution is unimodal and bell-shaped.
- Fifty per cent of the scores fall below the mean, and 50 per cent above the mean (that is, mean = median).
- Most of the scores pile up around the mean (i.e., mode = mean = median), and extreme scores (high or low) become increasingly rare (that is, have low frequencies).
- In theory, the left and right hand tails of the distribution touch the X-axis (that is, reach 0%) only at infinity. In practice, the frequencies of extreme values become so low that we expect to find less than 1 observation out of 10,000 with a value below the mean minus 4 times the standard deviation or with a value above the mean plus 4 times the standard deviation.

As soon as we know the mean and the standard deviation of a normal distribution, we know the complete distribution. (See Section 5.9 – Going further – for the formula to calculate the distribution on the basis of the mean and the standard deviation.) In addition, the interpretation of the mean and the standard deviation is quite straightforward, as shown in the next section.

4 See Chapter 3 for some notable exceptions – variables that are not normally distributed – such as the household sizes and the distribution of reaction times.

![Figure 5.3 The normal distribution. The shaded part shows the width of 1 standard deviation (σ).](image-url)
5.5.3 Two normal distributions with different means

A difference in means between two normal distributions implies that one distribution is shifted along the X-axis relative to the other distribution, as shown in Figure 5.4.

![Figure 5.4](image)

Figure 5.4 Two normal distributions with the same standard deviation but different means (relative frequency distributions).

A typical example of such a difference in means is the difference in height between 10-year old boys and 20-year old men. Both groups show more or less the same variability (that is, they have the same standard deviation), but the 10-year olds are some 38 cm smaller than the 20-year olds. According to the most recent American growth charts, this would translate into two normal distributions: one with a mean of 139 cm and a standard deviation of 6.5 cm for the 10-year olds, and the other with a mean of 177 cm and a standard deviation of 6.5 cm for the 20-year olds. This situation is shown in Figure 5.5. As you can see, the distance between both distributions is so big, that there is virtually no overlap of them. This agrees with our casual observations that 'all' 10-year old boys are smaller than 'all' 20-year old men.

![Figure 5.5](image)

Figure 5.5 The theoretical distributions of the heights of 10-year old boys ($\mu = 139, \sigma = 6.5$) and 20-year old men ($\mu = 177, \sigma = 6.5$).

5.5.4 Two normal distributions with different standard deviations

A change in the standard deviation alters the spread of the distribution (see Figure 5.6). A small standard deviation means that the data is more tightly clustered around the mean. A large standard deviation increases the spread of the data. Notice in Figure 5.6 that the narrower the bell shape, the higher it
Standardised scores, normal distribution and probability

becomes. This is because more observations are squeezed into a narrow range of observations.

A typical example of narrowing the standard deviation happens in the processing of fruit and vegetables. Have you ever wondered why all the tomatoes in your local supermarket are the same size – there is a small standard deviation of sizes? When the harvest was originally collected, there was a much bigger variation in the sizes and shapes of the crop. However, one of the first steps after the initial collection was to separate the fruits into different categories. The small and large tomatoes, together with the funny shaped ones, were discarded, for use in processed food. The remaining, medium-sized, well-shaped tomatoes were divided into separate classes with diameter differences of 1 cm at most, and only one size-range ended up in your supermarket.

The whole selection process was one of reducing the original, quite large range of sizes (going from tomatoes with a diameter of less than 2 cm to those with a diameter of more than 10 cm) to a very small range (of less than 1 cm). Whereas the tomatoes with a diameter between 5.5 cm and 6.5 cm might constitute only one-quarter of the original harvest, they may end up making the totality of the tomatoes you find in your grocery shop. As a result of the selection process a distribution with a large standard deviation (Figure 5.6(a)) has been reduced to a distribution with a small standard deviation (Figure 5.6(b)).

The difference in standard deviations between two normal distributions may be represented by changing the scale of the X-axis instead of the shape of the distribution. Figure 5.7 shows exactly the same situation as Figure 5.6, but this time the shapes of the two distributions have been kept constant and the scale of the X-axis has been altered.5

5 The fact that two different ways of representation can be used to show a change in standard deviation indicates that the height of the distribution is not important (i.e. the precise values on the Y-axis). What is important is the size of different areas in the curve (see the section on probability).
Applied to the tomato example, Figure 5.7(a) (unselected fruits) has a scale of diameters ranging from less than 2 cm to more than 10 cm. The scale in Figure 5.7(b) (for the selected fruits in your store) has a scale going from 5.5 cm to 6.5 cm.

The first way of representing a change in the standard deviation (with the thinner graph, as in Figure 5.6) shows us how the data become more piled around the mean when the standard deviation is smaller. (This will be important when we talk about the distribution of the means of samples in Chapter 7.)

The second way of representing a change in standard deviation (with the same graph, but a change of scale, as in Figure 5.7) tells us that we can use the same curve for every normal distribution, irrespective of the mean and the standard deviation. The only thing we have to do is adapt the scale of the X-axis.

For instance, Figure 5.8 shows the expected distributions of two tests, one with a mean of 100 and a standard deviation of 10, the other with a mean of 60 and a standard deviation of 6.

To draw a sketch like the curves in Figure 5.8, all you need to know are three things:
1. the general shape of the normal distribution;
2. where to place the mean of the distribution (at the vertical line that separates the distribution in two mirror halves);
3. where to place the SD (at 60% of the height of the vertical line).
These steps are illustrated in Figure 5.9, which shows a rapid sketch for a normal distribution with $M = 100$ and $SD = 10$.

(a) draw the shape of a normal distribution
(b) draw a vertical line through the centre to represent the mean
(c) draw a horizontal line at 60% of the height of the vertical line, then draw vertical lines to the X axis to represent $M + 1SD$ and $M – 1SD$
(d) add the values of $M + 2SD$ (estimated as twice the $SD$ from the previous step) and $M – 2SD$ (estimated as the mirror position of $M + 2SD$).

Figure 5.9 An illustration of the steps involved in making a rapid sketch of a normal distribution with $M = 100$ and $SD = 10$.

Make a sketch, as outlined in Figure 5.9, every time you are asked to interpret data from a test. This will allow you to rapidly and adequately position a score within the distribution (for example, where a score of 70 lies in a distribution with $M = 100$, $SD = 10$, and where it lies in a distribution with $M = 65$ and $SD = 2$).

As we will see in Section 5.0.0, it will also help you to get a fairly good estimate of the chances of observing lower and higher scores. These estimates will help you to avoid gross calculation errors when you are calculating the exact probabilities – a topic we will turn to in Sections 5.6 to 5.0.
5.6 Probability

In Section 5.7 we will see how z-scores and the assumption of a normal frequency distribution can be combined to make precise statements about the chances of observing lower and higher scores. Before we can do so, however, we need to introduce the concept of probability.

5.6.1 The definition of probability

Probability is an important concept in situations where several outcomes are possible. When more than one outcome is possible, we define the probability for any particular outcome as the frequency of that outcome divided by the frequency of all possible outcomes (that is, as the proportion of trials that resulted in the outcome).

For example, when you toss a coin, there are two possible outcomes: heads or tails. You can estimate the probability of the outcome 'heads' by tossing the coin say 100 times and count how many of these trials result in heads. So,

\[
p(\text{heads}) = \frac{\text{Number of outcomes which are heads}}{\text{Total number of trials}}
\]

Suppose we find that 60 of the 100 trials result in the outcome 'heads' and 40 result in the outcome 'tails'. Then, the probability of heads is:

\[
p(\text{heads}) = \frac{60}{60 + 40} = 0.60
\]

and the probability of tails is:

\[
p(\text{tails}) = \frac{40}{60 + 40} = 0.40
\]

The probabilities add up to 1.00 (or 100%) because there are only two possible outcomes.

5.6.2 How to calculate a probability from a frequency distribution table

The probability of an outcome is easy to calculate when we have a frequency distribution table. In that case, we simply divide the frequency of the outcome by the sum of all the observations. Thus:

\[
p(X) = \frac{f(x)}{N}
\]

For instance, suppose the frequency distribution table represents the gross annual household incomes in a town with 25,797 households (Table 5.1).

On the basis of Table 5.1, it is not difficult to calculate the probability of a household in the £15,000–20,000 range. This probability is defined as:

\[
p(15K \leq X < 20K) = \frac{9541}{25797} = 0.37
\]

We may have expected that half of the trials would result in heads because, for an unbiased coin, \( p(\text{heads}) = 0.50 \). However, this is an experimental result. We can use statistics to see the likelihood of our observed value (\( p = 0.60 \)) and whether this deviation from the expected value is likely to be due to chance fluctuations –sampling error– or should be interpreted as an indication that our coin was not well balanced or that we did not toss well.
### 5.6.3 How to calculate a probability from a frequency distribution graph

Probabilities can also be obtained from a frequency distribution graph. To do this, we calculate the area of the histogram that corresponds to the $X$-values of interest, and divide this by the total area. Figure 5.10 has been plotted from the

![Figure 5.10](image-url)
family income data above. So, if we want to know the probability of an annual gross income less than £15,000 a year on the basis of the frequency distribution graph shown in Figure 5.10, we calculate the area of the first three bars (shaded) and we divide this by the area of all the bars.

To calculate the area of a bar in a frequency distribution graph, measure the height of each bar. In Figure 5.10, the height of the first bar is 1 mm; that of the second bar is 6 mm; that of the third bar is 9 mm and so on until you get the following table:

<table>
<thead>
<tr>
<th>Income band</th>
<th>Height of the bar in the graph (in mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X &lt; 5K$</td>
<td>1</td>
</tr>
<tr>
<td>$5K \leq X &lt; 10K$</td>
<td>6</td>
</tr>
<tr>
<td>$10K \leq X &lt; 15K$</td>
<td>9</td>
</tr>
<tr>
<td>$15K \leq X &lt; 20K$</td>
<td>55</td>
</tr>
<tr>
<td>$20K \leq X &lt; 25K$</td>
<td>51</td>
</tr>
<tr>
<td>$25K \leq X &lt; 30K$</td>
<td>14</td>
</tr>
<tr>
<td>$30K \leq X &lt; 35K$</td>
<td>6</td>
</tr>
<tr>
<td>$35K \leq X &lt; 40K$</td>
<td>4</td>
</tr>
<tr>
<td>$40K \leq X$</td>
<td>0.5</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>146.5</td>
</tr>
</tbody>
</table>

Because each bar has the same width, we can use the height of each bar as a measure of its area. The summed area of the first three bars is $1 + 6 + 9 = 16$. The area of all the bars is 146.5. So, the probability of observing a gross annual income of less than £15K equals $16/146.5 \approx 11\%$. Check this outcome with the exact outcome you would obtain on the basis of the frequency distribution table.

5.6.4 How to calculate a probability from a relative frequency distribution table

Probabilities have a very transparent relationship to relative frequencies expressed as percentages (see Chapter 2): if you multiply a probability by 100, you get the relative frequency. So, in our example of gross annual incomes, a probability of .37 that a household has an income between £15K and £20K means that 37% of the households have an income of this size. Conversely, by dividing the relative frequencies expressed as percentages by 100, we get the associated probabilities. This is shown in Table 5.3, which includes the raw frequencies, as well as the relative frequencies and the probabilities of the household incomes from our example.

So, calculating probabilities on the basis of a frequency distribution table with relative frequencies is very easy. You just divide the percentages by 100,
and you get the associated probabilities. The relationship between probability and relative frequency is one of the reasons why many researchers present their data as relative frequencies rather than as raw frequencies (see Chapter 2).

### 5.6.5 How to calculate a probability from a relative frequency distribution graph

There are two ways in which we can calculate probabilities when we have a relative frequency distribution graph. We could calculate the areas of the bars, as discussed in Section 5.6.3, or we could try to estimate the individual relative frequencies from the graph. When we choose for the latter option, we are basically trying to reconstruct the relative frequency distribution table. Figure 5.11

---

**Table 5.3 Relationship between relative frequency and probability**

<table>
<thead>
<tr>
<th>Income band</th>
<th>( f )</th>
<th>Relative frequency (%)</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X &lt; 5K )</td>
<td>230</td>
<td>0.9%</td>
<td>0.009</td>
</tr>
<tr>
<td>( 5K \leq X &lt; 10K )</td>
<td>890</td>
<td>3.4%</td>
<td>0.034</td>
</tr>
<tr>
<td>( 10K \leq X &lt; 15K )</td>
<td>1620</td>
<td>6.3%</td>
<td>0.063</td>
</tr>
<tr>
<td>( 15K \leq X &lt; 20K )</td>
<td>9541</td>
<td>37.0%</td>
<td>0.370</td>
</tr>
<tr>
<td>( 20K \leq X &lt; 25K )</td>
<td>8954</td>
<td>34.7%</td>
<td>0.347</td>
</tr>
<tr>
<td>( 25K \leq X &lt; 30K )</td>
<td>2543</td>
<td>9.9%</td>
<td>0.099</td>
</tr>
<tr>
<td>( 30K \leq X &lt; 35K )</td>
<td>1166</td>
<td>4.5%</td>
<td>0.045</td>
</tr>
<tr>
<td>( 35K \leq X &lt; 40K )</td>
<td>754</td>
<td>2.9%</td>
<td>0.029</td>
</tr>
<tr>
<td>( 40K \leq X )</td>
<td>99</td>
<td>0.4%</td>
<td>0.004</td>
</tr>
<tr>
<td>( \Sigma = 25,797 )</td>
<td></td>
<td>100.0%</td>
<td>1.000</td>
</tr>
</tbody>
</table>

---

**Figure 5.11** Probability calculated on the basis of a relative frequency distribution graph
illustrates how we can use a relative frequency distribution graph to estimate the probability of an income below £15K. Now, do the same to estimate the probability of a gross annual income between £30K and £40K.

The probability of an income lower than 15K equals the sum of the estimated probabilities associated with the first three bars (shaded). Roughly, the sum of the relative frequencies in the grey area will agree with 1% (first bar) + 4% (second bar) + 6% (third bar) = 11%, which will give a probability of $p = .11$.

5.7 Z-scores, normal distributions, probabilities and percentiles

5.7.1 The standard normal distribution

In this section, we will combine the three concepts we have introduced thus far: z-scores, the normal distribution, and probabilities. We will find out how z-scores can be used to obtain more precise information about the position of an individual within a population. Remember that, in Sections 5.2 to 5.4 on z-scores, we limited ourselves to deciding whether an individual belonged to the upper or the lower half of the distribution (+ or − sign) and to calculating the typical (less than 1 SD from the mean) or the exceptional/abnormal range (more than 2 SD). Now, we will use our knowledge about the normal distribution to make more precise statements.

First, we need to understand the concept of the standard normal distribution. This is a theoretical normal frequency distribution with mean = 0 and standard deviation = 1. Because the mean and the standard deviation fully define a normal distribution, we can draw the standard normal distribution. We just use the sketches from Figure 5.9, and insert a mean value of $M = 0$, and a standard deviation of $SD = 1$ (Figure 5.12).

![Figure 5.12 A quick sketch of the standard normal distribution.](image)

On the basis of the equation used for the normal distribution, we can calculate exactly how much of the area of the curve is situated in the different parts of the curve. In the same way as we used the frequency distribution graphs in Figures 5.10 and 5.11 to determine how many households had incomes in a certain range, we can use our knowledge about the normal distribution to determine the properties of our data. For example, we might want to know how many observations are expected to be below a standard normal score of −2 (that is, two standard deviations below the mean). This probability is given by dividing the area of the small part of the curve below −2 by the area of the complete
distribution. If we do so, we obtain a value of $p = .0228$, meaning that 2.28% of all observations in the standard normal distribution are expected to have a value of less than –2. Similarly, we can calculate the probability of a standard normal score between –2 and –1. This probability equals .1359, meaning that 13.59% of the observed standard normal scores are expected to lie in this range. In fact, we do not need to make any estimates of area because these areas are already known and are shown in Figure 5.13.

Figure 5.13 shows that 34% of the area of the distribution lies between a standard normal score of 0 (the mean) and a score of +1 (one standard deviation above the mean). Similarly, another 34% of the area lies between 0 and –1. This means that scores between –1 and +1 are expected to occur with a probability of 68% (thus, slightly more than two-thirds of the probability of all possible outcomes). This is called the typical range. Similarly, Figure 5.13 shows that $X$-values larger than +2 are expected to occur in less than 2.5% of the outcomes; the same is true for values less than –2. We have called this the exceptional range, because values more extreme than –2 or +2 are expected in less than 5% of all observations.

Figure 5.13 also shows how we can use the standard normal distribution to extract information about the exact position of an individual relative to all other individuals. Suppose an individual has a standardised score of +1 on a test. Figure 5.13 informs us that the chance of finding a higher score is 2.25% + 13.59% = 15.84%. The chance of finding a lower score is 2.28% + 13.59% + 34.13% + 34.13% = 84.13%. Thus, if this score represented the result of an exam, the individual would belong to the top 20% of students.

5.7.2 The relationship of the standard normal distribution to other normal distributions

You may already have figured out that the standard normal distribution, as depicted in Figure 5.13, is nothing more than a normal distribution of $z$-scores. This means that we can reduce every possible normal distribution to the standard normal distribution by using $z$-scores instead of the original raw scores. Similarly, we can replace the standard normal distribution by any other normal distribution by converting the $z$-scores into raw scores. The only thing we have to do is to change the scale of the $X$-axis (Figures 5.7 and 5.8).
For instance, many IQ tests yield a normal distribution of scores and have been constructed so that their mean is 100 and their standard deviation is 15. According to these tests, people with an IQ between 85 and 115 have an average intelligence; people with an IQ below 70 and above 130 are called ‘exceptional’ (in the former case ‘retarded’, in the latter case ‘gifted’). About two-thirds of people (68%) are expected to have IQs between 85 and 115; roughly 2.5% are expected to have an IQ lower than 70, and 2.5% are expected to have an IQ higher than 130. Figure 5.14 shows the application of the normal distribution to IQ scores. Notice that the only thing that has changed from Figure 5.12 (the standard normal distribution) is the values on the X-axis.

![Figure 5.14](image)

**Figure 5.14** Probabilities associated with different parts of the distribution of IQ scores, when the test results in a normal distribution with \( M = 100 \) and \( SD = 15 \).

Before we move on to calculate exact probabilities on the basis of the standard normal distribution, there is one cautionary note: the standard normal distribution can only be used to calculate probabilities of data that come from a normal frequency distribution. Changing a raw score into a z-score does not change the shape of the distribution; it only changes the values of the X-axis. Thus, if the original distribution was bimodal or positively skewed, the z-scores will follow the same distribution, and probabilities calculated on the basis of the standard normal distribution will yield bad estimates.

The standard normal distribution is used in many cases because we can assume that most variables follow a normal distribution (see Chapter 6). However, if you have any reason to doubt whether your data are normally distributed, always check the frequency distribution of your raw data (as discussed in Chapter 2) before going any further.

### 5.7.3 Calculating exact probabilities using the standard normal distribution

In the preceding section, we have calculated the probabilities associated with the standard normal distribution on the basis of a crude graph (Figure 5.13) that
provided us with the probabilities of the z-scores –2, –1, 0, 1, and 2. We could create a figure that gives us the probabilities for z-scores that are multiples of .5, and that would allow us to make more precise calculations. Unfortunately, such a figure would become quite cluttered and difficult to read. Therefore, rather than keep on using diagrams, we will turn to a table. Table 5.4 contains exactly the same information as we would have in our graph of z-scores at intervals of .5. All the information displayed in Figure 5.13 is presented in Table 5.4, as well as the intermediate scores (for example, the probability of observing a z-score lower than –2 is .0228).

Table 5.4 Probabilities of lower and higher scores associated with z-scores that are multiples of .5.

<table>
<thead>
<tr>
<th>z-score</th>
<th>p(lower score)</th>
<th>p(higher score)</th>
</tr>
</thead>
<tbody>
<tr>
<td>–3.0</td>
<td>.0013</td>
<td>.9987</td>
</tr>
<tr>
<td>–2.5</td>
<td>.0062</td>
<td>.9938</td>
</tr>
<tr>
<td>–2.0</td>
<td>.0228</td>
<td>.9772</td>
</tr>
<tr>
<td>–1.5</td>
<td>.0668</td>
<td>.9332</td>
</tr>
<tr>
<td>–1.0</td>
<td>.1587</td>
<td>.8413</td>
</tr>
<tr>
<td>–0.5</td>
<td>.3085</td>
<td>.6915</td>
</tr>
<tr>
<td>0.0</td>
<td>.5000</td>
<td>.5000</td>
</tr>
<tr>
<td>+0.5</td>
<td>.6915</td>
<td>.3085</td>
</tr>
<tr>
<td>+1.0</td>
<td>.8413</td>
<td>.1587</td>
</tr>
<tr>
<td>+1.5</td>
<td>.9332</td>
<td>.0668</td>
</tr>
<tr>
<td>+2.0</td>
<td>.9772</td>
<td>.0228</td>
</tr>
<tr>
<td>+2.5</td>
<td>.9938</td>
<td>.0062</td>
</tr>
<tr>
<td>+3.0</td>
<td>.9987</td>
<td>.0013</td>
</tr>
</tbody>
</table>

Notice that the lower half of Table 5.4 is a mirror image of the upper half. As an example, compare the probabilities of \( z = –3.0 \) and \( z = +3.0 \). This is to be expected, given that the normal distribution is a symmetrical distribution (see Figure 5.13). Also, notice that the column of \( p(\text{higher score}) \) always equals \( 1 – p(\text{lower score}) \). Why is this?

Table 5.4 allows us to calculate exact probabilities associated with z-scores that are multiples of .5. Of course, nothing prevents us from making a larger table with the probabilities of all z-scores between –3.0 and +3.0 with a precision of .01. Indeed, this table is presented in Appendix A. Using Appendix A, therefore, we can calculate the exact probabilities associated with the standard normal distribution.

For reasons that will become clear in the next chapter, two other important z-values are \( z = –1.96 \) and \( z = +1.96 \). For these values, the chances of observing more extreme data are exactly .025. Thus, \( p(\text{lower value}) \) of \( z = –1.96 \) is .025 and \( p(\text{higher value}) \) of \( z = +1.96 \) is .025 as well. This is shown in Figure 5.15.
It is important for you to understand that Appendix A is just a more detailed version of the data in Figure 5.13. When you are asked to calculate probabilities on the basis of Appendix A, you are strongly advised first to draw a rough sketch of the normal distribution (see Figure 5.16) and to make a rough estimate of the probabilities on the basis of your knowledge of the standard normal distribution. In that way, you will never make gross calculation errors, because you already have a good idea of the values you should obtain. For instance, when you are asked to look up the exact probabilities associated with \( z = +0.23 \), you know beforehand that this value lies slightly above a \( z \)-value of . You know that \( p(\text{lower}) = .50 \) and \( p(\text{higher}) = .50 \) at a \( z \)-value of 0, because this \( z \)-value is the median of the standard normal distribution. So, even without consulting Appendix A, you already know that, for \( z = +0.23 \), \( p(\text{lower}) \) will be slightly more
than .50, and \( p(\text{higher}) \) will be slightly less than .50. The only thing to do then is to look up the exact values!

Because we can translate all normal frequency distributions to the standard normal distribution by calculating \( z \)-scores, we can use Appendix A to calculate the probabilities for each and every normal distribution. So, we can calculate the probability of observing an IQ-score lower than 92 on a test with \( M = 100 \) and \( SD = 15 \). First, we make an approximate sketch of the situation (Figure 5.17), which informs us that an IQ-score of 92 lies nearly half a standard deviation below the mean. Thus the \( z \)-score will be roughly -0.50, \( p(\text{lower}) \) will be less than .50, and \( p(\text{higher}) \) will be more than .50. When we then do the exact calculations using Appendix A, we obtain \( z = (92 - 100)/15 = -0.53; \) \( p(\text{lower}) = .2981; \) \( p(\text{higher}) = .7019 \) (see also Table 5.4).

**5.7.4 Percentiles and percentile ranks**

In Section 5.7.3, we always talked about \( p(\text{lower}) \) and \( p(\text{higher}) \) to indicate the probability of observing scores lower and higher than a given \( X \)-value. Because these are rather awkward terms for everyday communication, statisticians have invented the term ‘percentile rank’. Percentile rank is defined as the percentage of individuals in the distribution that have a score below the \( X \)-value on which the rank is calculated. So, it is just \( 100 \times p(\text{lower}) \). Applied to our examples above, the percentile rank of the \( z \)-score .23 in Figure 5.16 is \( 100 \times .591 = 59 \), indicating that 59% of the individuals are expected to have a \( z \)-score below .23. Similarly, the percentile rank of the IQ-score of 92 in Figure 5.17 is \( 100 \times .2981 = 30 \), indicating that 30% of the people are expected to have an IQ-score below 92.

Put simply, a person is doing better than average if they have a percentile rank above 50. Similarly, their performance is below average when they have a percentile rank of less than 50. Therefore, it is much easier to talk to the parents of a child with a percentile rank of 90 on an IQ test than the parents of a child with a percentile rank of 10.

**Learning check 3**

Find out, with the help of Appendix A, which IQ scores on a test with \( M = 100 \) and \( SD = 15 \) correspond to the percentile ranks 10 and 90? (See the Learning check solutions at the end of the chapter).
A percentile is the $X$-value that corresponds to a particular percentile rank or $p(\text{lower})$. So, if we have an IQ test that yields normally distributed scores with $M = 100$ and $SD = 15$, which score is the 95th percentile of this test? To answer this question, it may be good to consider that the 95th percentile is a very high score. It puts the person in the top 5% because 95% of the individuals taking the test will do worse. So, the score will be higher than the mean score. From our knowledge of the normal distribution (Figure 5.13), we know that it will be quite close to the mean plus two standard deviations (which would yield a percentile of 98). A look at Appendix A tells us that percentile 95 agrees with a $z$-score of +1.65 (which has a $p(\text{lower})$ of .9505 and a $p(\text{higher})$ of .0495). Then we use our knowledge from Section 5.4 to calculate the IQ score that agrees with a $z$-score of +1.65. This score is $M + 1.65 \times SD = 100 + 1.65 \times 15 = 125$.

As was the case for percentile ranks, percentiles above 50 refer to $X$-values above the mean and percentiles below 50 refer to $X$-values below the mean. A percentile of 2 will agree with an $X$-value close to $M - 2SD$, and a percentile of 98 will agree with an $X$-value of $M + 2SD$.

Because we have covered quite some ground in the preceding pages, the next section is a cookbook summary of how to calculate probabilities on the basis of the standard normal distribution.

### 5.7.5 Step by step: determining probabilities and percentile ranks associated with a normal distribution

Mr. Wilcock obtained a score of 26 on a test with mean 40 and standard deviation 12 (scores are normally distributed). Calculate his $z$-score, the probability of obtaining a lower score, the probability of obtaining a higher score, the percentile rank, and the percentile this score represents.

**Step 1** Sketch the distribution of the test scores and shade the part below Mr. Wilcock’s score. Use the probabilities from Figure 5.13 to get a rough estimate of the different measures.

![Distribution sketch](image)

From the sketch we know that Mr Wilcock performed below the mean, even below one standard deviation under the mean. So, the $z$-score will be less than –1.0, $p(\text{lower})$ will be less than .16 (2% + 14%), and $p(\text{higher})$ will be more than .84. Similarly, the percentile rank will be lower than 16, and Mr Wilcock’s score will be lower than percentile 16. Now that we have these rough estimates, we are in a better shape to calculate the exact figures.
Step 2  Transform the \( X \)-value into a \( z \)-score.

\[
z = \frac{X - M}{SD}
\]

in which \( M \) = the mean of the test and \( SD \) = the standard deviation.

Applied to the example:

\[
z = \frac{26 - 40}{12} = \frac{-14}{12} = -1.17
\]

Step 3  Determine the probability of a lower score \([p(\text{lower})]\) and the probability of a higher score \([p(\text{higher})]\).

Go to Appendix A. Look under the entry for \( z = -1.17 \). There you find the two values you are looking for: \( p(\text{lower}) = .1210 \), and \( p(\text{higher}) = .8790 \). That is, the proportion of the standard normal distribution lower than \( z = -1.17 \) amounts to .1210; and the proportion of the distribution higher than \( z = -1.17 \) amounts to .8790.

Step 4  The percentile rank is the percentage of the distribution below the \( z \)-score. This is obtained by multiplying \( p(\text{lower}) \) by 100. So, the percentile rank of Mr Wilcock's score is \( .1210 \times 100 = 12 \), meaning that only 12% of the population is expected to have a score on the test lower than Mr Wilcock.

Step 5  Because Mr Wilcock’s percentile rank is 12, his score of 26 on the test corresponds to percentile 12.

Performance checks

1  Negative \( z \)-scores imply that the person performed below the mean; positive \( z \)-scores indicate that the person performed better than average. Is this the case?

2  Percentile ranks of negative \( z \)-scores are always lower than 50; those of positive \( z \)-scores are higher than 50 (the percentile rank of \( z = 0.0 \) is 50).

3  Look at the sketch you made under Step 1 and check whether the exact numbers you calculated in Steps 2–5 roughly agree with those initial estimates. If not, try to find out what caused the difference. This type of performance check will ensure that you have a good overview of the total situation, and that your calculations are never grossly wrong.
5.8 Calculating $z$-scores and probabilities in SPSS

To illustrate how we can calculate $z$-scores in SPSS, we will use the example of the number of pints of beer drunk by people in the previous weekend (Section 3.2):

5, 4, 3, 5, 4, 2, 4, 3, 6

To find the $z$-scores, we click on Analyze, Descriptive Statistics, and Descriptives.

We have to indicate the variable for which we want the descriptives, so we tick the checkbox Save standardized values as variables.
By checking this box, a column will be added to your data file with the z-scores. Click on OK and look at your data screen. This is what you should get.

To calculate the probability of lower z-scores (that is, the part of the curve to the left of the observed z-score), click on Transform, Compute Variable.
This will give you the following screen:
Here you have to do two things. First, enter a name in the section **Target Variable** (Prob_lower) ①. Then you have to go under the heading **Function group** and **CDF and Noncentral CDF** ② (CDF stands for cumulative density function). Select **Cdf Normal** ③ from the alphabetical list.

Then put your cursor in the panel **Numeric Expression** ④ (position your mouse in this panel and click once). Then click on the upward arrow to bring the Cdf.Normal function to the panel ⑤.
Standardised scores, normal distribution and probability

CDF \text{NORMAL}(\text{quant}, \text{mean}, \text{stdev}). \text{Numeric.} \text{Returns the cumulative probability that a value from the normal distribution, with specified mean and standard deviation, will be less than quant.}
There are three values to be entered now. The first is your z-score. You enter this by clicking on Zscore(Pints) ① and then on the arrow pointing to the panel Numeric Expression.

If everything went well, this is what you should have:
The next two values to enter are the mean and the standard deviation. Because we are working with z-scores, they are easy: 0 and 1.

If you click on OK, you will see that a new column has been added to your data called Prob_lower. It gives you \( p(\text{lower}) \) for each z-value (and, hence, each number of pints). Check a few values with Appendix A!

This allows you to rapidly calculate the probabilities associated with your values.

### 5.9 Going further: defining the shape of the normal distribution

Although we could digress onto many interesting topics (such as how the exact values of \( p(\text{lower}) \) are calculated in SPSS), we will deal with only one of them here – how the shape of the normal distribution is defined. It is based on the equation:

\[
f(X) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(X-\mu)^2}{2\sigma^2}}
\]

If you look carefully, you see that this expression is an exponential function of the form \( f(X) = a^e b \). Because the exponent is a negative squared value, \( f(X) \) will always be negative unless \( X = \mu \). So, the function will have its maximum when
X = μ and will tend to 0 when X << μ or X >> μ. The rest of the equation simply contains variables and constants to make sure the area under the distribution equals 1.

5.10 Answers to chapter questions

1 What is a standardised score and why is this helpful?
A standardised score is a score expressed as the number of standard deviations from the mean. So, a standardised score of 0 means that the raw score coincides with the mean of the distribution it is 0 standard deviations away from the mean. A standardised score of –1 means that the raw score is 1 standard deviation below the mean. The standardised score is helpful because people immediately know how to interpret it and because it allows psychologists to directly compare the scores of different tests.

2 What is a z-score?
A z-score is another name for a standardised score.

3 How do we transform an original score X into a z-score?

\[ z = \frac{X - M}{SD} \]

where M = the mean of the distribution and SD = the standard deviation of the distribution.

4 What is the rough interpretation of the value of a z-score? What does a z-score of 0 stand for?
A z-score lower than –2 represents a very bad performance, a z-score higher than +2 represents a very good performance, and a z-score between –1 and +1 represents an average performance. A z-score of 0 means that the score is exactly in the middle of the distribution.

5 How do we transform z-scores in raw X-scores?

\[ X = M + z \times SD. \]

6 What is the normal distribution?
A normal distribution is a distribution that looks as follows:

The function is fully defined by the mean (μ) and the standard deviation (σ). Many observations and measurements of real life features are distributed like this. The normal distribution forms the basis of all parametric statistics.
7 What is the probability of an outcome?
The probability of any particular outcome is the frequency of this outcome divided by the sum of the frequencies of all possible outcomes.

8 How do we calculate the probability?
\[ p(X) = \frac{f(X)}{N} \], in which \( f(X) \) and \( N \) are either obtained from a frequency distribution table or estimated on the basis of a frequency distribution graph.

9 What is the standard normal distribution and why do we use it?
The standard normal distribution is a normal distribution with \( \mu = 0 \) and \( \sigma = 1 \). It is the distribution of the \( z \)-scores, assuming the original \( X \)-scores formed a normal distribution. We use the standard normal distribution to calculate the probability values (p-values) associated with \( X \)-scores. It is a development of the normal distribution and is, therefore, also the function on which parametric statistics are based.

10 How do we calculate the probability associated with a \( z \)-score?
There are several methods. First, we can get a good idea by making use of a sketch as outlined in Figure 5.13. Second, we can make use of Appendix A if we want to have precise values. Finally, we could write a function in Excel that will give us the probability associated with a \( z \)-score.

11 What is a percentile rank and what is a percentile?
A percentile rank is the percentage of individuals in the distribution that have a score below the \( X \)-value on which the rank is calculated. A percentile is the \( X \)-value that corresponds to a particular percentile rank or \( p(\text{lower}) \).

5.11 Learning check solutions

Learning check 1
\( z \)-score Mr Parkhurst: \( z = (35 - 40)/10 = -5/10 = -0.5 \)

Learning check 2
\( p(\text{lower}) \) and \( p(\text{higher}) \) for \( z = +0.23 \) are \( p(\text{lower}) = .5910 \) and \( p(\text{higher}) = .4090 \). Looking at Figure 5.13, you can see that \( z = +0.23 \) lies slightly above \( z = .00 \), at which point \( p(\text{lower}) = .5 \) and \( p(\text{higher}) = .50 \). So, \( p(\text{lower}) \) should be slightly higher than .50, and \( p(\text{higher}) \) should be slightly lower.

Learning check 3
A percentile rank of 10 means that \( p(\text{lower}) = .10 \). Appendix A informs us that we will find this at \( z = -1.28 \). A \( z \)-value of -1.28 corresponds to an IQ score of \( 100 + (-1.28 \times 5) = 100 - 19 = 81 \). Similarly, a percentile rank of 90 means that \( p(\text{lower}) = .90 \), which we will find at \( z = +1.28 \), corresponding to IQ = \( 100 + (128 \times 15) = 119 \).