CHAPTER 11

Matrices

B5 Applications in control engineering

Next we examine applications of matrices in control engineering. We use the following definitions of control:

Let **A**, **B** and **C** be matrices, where **A** is a 2×2 matrix, **B** is a 2×1 matrix and **C** is a 1×2 matrix. That is **A** is a square matrix, **B** is a column vector and **C** is a row vector.

The controllability matrix, $\mathbf{C}_T$, is defined as

11.6 $$\mathbf{C}_T = (\mathbf{B} \quad \mathbf{AB})$$

where **B** forms the first column and the product **AB** forms the second column of the matrix $\mathbf{C}_T$.

The observability matrix, $\mathbf{O}_T$, is defined as

11.7 $$\mathbf{O}_T = \begin{pmatrix} \mathbf{C} \\ \mathbf{CA} \end{pmatrix}$$

where **C** forms the first row and the product **CA** forms the second row of $\mathbf{O}_T$.

Example 14 *control engineering*

The state–space equations of a system has the following matrices:

$$\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{C} = (2 \quad 3)$$

Find $\mathbf{C}_T$ and $\mathbf{O}_T$.

Solution

First we obtain the product **AB**:

$$\mathbf{AB} = \begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} (-1 \times 1) + (1 \times 1) \\ (2 \times 1) + (-3 \times 1) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Substituting $\mathbf{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{AB} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ into $\mathbf{C}_T = (\mathbf{B} \quad \mathbf{AB})$ gives

$$\mathbf{C}_T = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

Example 14 *continued*

To find the observability matrix, $\mathbf{O}_T$, we first obtain the product **CA**:

$$\mathbf{CA} = (2 \quad 3)\begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix} \big([2 \times (-1)] + (3 \times 2) \qquad (2 \times 1) + [3 \times (-3)]\big) = (4 \quad -7)$$

Using $\mathbf{O}_T = \begin{pmatrix} \mathbf{C} \\ \mathbf{CA} \end{pmatrix}$ with $\mathbf{C} = (2 \quad 3)$ and $\mathbf{CA} = (4 \quad -7)$ gives $\mathbf{O}_T = \begin{pmatrix} 2 & 3 \\ 4 & -7 \end{pmatrix}$

Example 15 *control engineering*

The system poles are defined to be those values of s which satisfy

$$\det(s\mathbf{I} - \mathbf{A}) = 0$$

These s are also called the eigenvalues of **A**. We will discuss eigenvalues in **Section F**. Obtain the system poles for

$$\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 2 & -1 \end{pmatrix}$$

Solution

Remember that **I** is the identity matrix. Therefore we have

$$s\mathbf{I} - \mathbf{A} = s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & -4 \\ 2 & -1 \end{pmatrix} \qquad \left[\text{Because } \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$$= \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 3 & -4 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} s-3 & 0-(-4) \\ 0-2 & s-(-1) \end{pmatrix} = \begin{pmatrix} s-3 & 4 \\ -2 & s+1 \end{pmatrix}$$

Hence

$$\det(s\mathbf{I} - \mathbf{A}) = \det\begin{pmatrix} s-3 & 4 \\ -2 & s+1 \end{pmatrix}$$

$$= (s-3)(s+1) - (-2 \times 4) \qquad [\text{By } 11.1]$$

$$= s^2 - 2s - 3 + 8$$

$$= s^2 - 2s + 5$$

We need to solve the quadratic equation, $s^2 - 2s + 5 = 0$, to find the system poles. Using the quadratic formula we have

$$s = \frac{2 \pm \sqrt{(-2)^2 - (4 \times 1 \times 5)}}{2} = \frac{2 \pm \sqrt{-16}}{2}$$

11.1 $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$

Example 15 *continued*

How do we find $\sqrt{-16}$?

We need to use complex numbers: $\sqrt{-16} = j4$

$$s = \frac{2 \pm j4}{2} = \frac{2}{2} \pm j\frac{4}{2} = 1 \pm j2$$

The system poles are at $s = 1 + j2$ or $s = 1 - j2$

SUMMARY

In control engineering there are a number of formulae which can be used to find matrices such as controllability and observability matrices. We can also use matrices to find the system poles.

Exercise **11(b)**

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

Questions 9 to 15, inclusive, are in the field of [*control engineering*].

9 By using 11.6 and 11.7 determine the controllability matrix, $\mathbf{C}_T$, and the observability matrix, $\mathbf{O}_T$, for

$$\mathbf{A} = \begin{pmatrix} -5 & 6 \\ 2 & 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \text{ and } \mathbf{C} = (2 \quad 9)$$

10 By using 11.6 and 11.7 find the controllability matrix, $\mathbf{C}_T$, and the observability matrix, $\mathbf{O}_T$, for the system with

$$\mathbf{A} = \begin{pmatrix} 2 & 5 \\ -1 & 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{C} = (-1 \quad 4)$$

A system is controllable if $\det\mathbf{C}_T \neq 0$ and is observable if $\det\mathbf{O}_T \neq 0$. Determine whether the above system is

i controllable **ii** observable

11 By using definitions of controllable and observable of question **10**, check whether the following systems are controllable and/or observable:

a $\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 9 & 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ and $\mathbf{C} = (3 \quad 0)$

b $\mathbf{A} = \begin{pmatrix} 6 & 2 \\ 1 & 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{C} = (3 \quad 2)$

For questions 12 to 15 the system poles are defined in Example 15 (those values of s which satisfy $\det(s\mathbf{I} - \mathbf{A}) = 0$).

12 Determine the system poles for

a $\mathbf{A} = \begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 2 & 10 \\ 5 & -3 \end{pmatrix}$

c $\mathbf{A} = \begin{pmatrix} -5.1 & 2.2 \\ 3.7 & 6.1 \end{pmatrix}$

13 The state matrix, **A**, of an *RLC* series circuit is given by

$$\mathbf{A} = \begin{pmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{pmatrix}$$

Determine the system poles of **A**.

Exercise 11(b) continued

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

14 For a mechanical system the state matrix, **A**, is given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{pmatrix}$$

Find the system poles for **A**.

15 If the real part of the system poles, s, is negative, that is

$$\text{Re}(s) < 0$$

then the system is said to be stable. Which of the following systems are stable?

a $\mathbf{A} = \begin{pmatrix} -1 & -3 \\ 5 & 2 \end{pmatrix}$

b $\mathbf{A} = \begin{pmatrix} -6 & -1 \\ 9 & -3 \end{pmatrix}$

c $\mathbf{A} = \begin{pmatrix} 3.1 & 6.5 \\ 1.7 & 4.8 \end{pmatrix}$

SECTION H Applications in heat transfer

By the end of this section you will be able to:

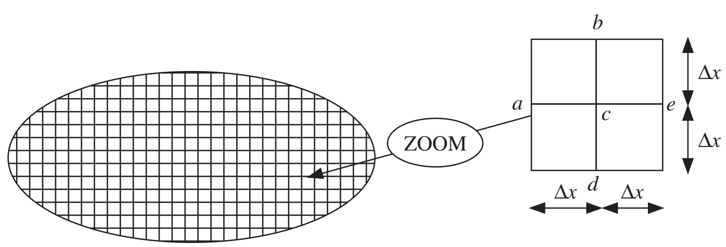
- form nodal equations
- solve the resulting equations by using a graphical calculator and a computer algebra system

The techniques described in this section are used in many engineering disciplines such as electric field, hydrodynamics and finite element analysis. However in this section we concentrate on the application to heat transfer.

H1 Applications of matrices in heat transfer

Figure 22 shows a body divided up into squares of size Δx.

Fig. 22



The following formula gives the average temperature, T_c, close to a node (point) c in the body:

11.21 $$T_a + T_b + T_d + T_e = 4T_c$$

where T_a, T_b, T_d and T_e represent temperature at nodes a, b, d and e respectively.

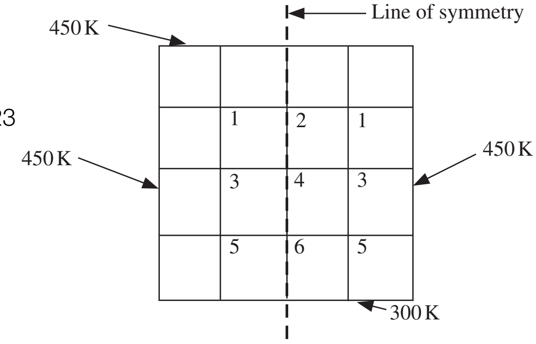
The point c is an interior point of the body surrounded by the points a, b, d and e. Each of these points is Δx away from c.

11.21 is an equation used in heat transfer. We will not derive this equation but just apply it in the appropriate case.

Example 42

Figure 23 shows a square cross-section with the stated surface temperatures. Find the temperatures at the interior nodes 1, 2, 3, 4, 5 and 6.

Fig. 23

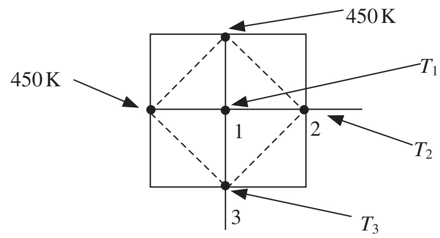


Solution

Let T_1, T_2, T_3, T_4, T_5 and T_6 represent the temperatures at nodes 1, 2, 3, 4, 5 and 6 respectively.

Consider node 1. We have the situation shown in Fig. 24.

Fig. 24



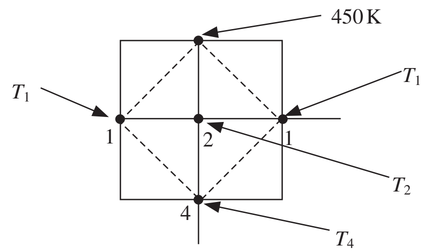
Applying 11.21 to a node means we sum the four surrounding temperatures, that is

$$450 + 450 + T_2 + T_3 = 4T_1$$

Similarly for node 2 (Fig. 25).

Examining Figure 25 and applying 11.21 we have

Fig. 25



$$T_1 + 450 + T_1 + T_4 = 4T_2$$

Following this procedure for the remaining nodes we obtain:

Node 1: $450 + 450 + T_2 + T_3 = 4T_1$
Node 2: $T_1 + 450 + T_1 + T_4 = 4T_2$
Node 3: $450 + T_1 + T_4 + T_5 = 4T_3$
Node 4: $T_3 + T_2 + T_3 + T_6 = 4T_4$
Node 5: $450 + T_3 + T_6 + 300 = 4T_5$
Node 6: $T_5 + T_4 + T_5 + 300 = 4T_6$

Simplifying and rearranging these equations yields

$$\begin{array}{rrrrrrl}
4T_1 & -T_2 & -T_3 & & & & = 900 \\
-2T_1 & +4T_2 & & -T_4 & & & = 450 \\
-T_1 & & +4T_3 & -T_4 & -T_5 & & = 450 \\
& -T_2 & -2T_3 & +4T_4 & & -T_6 & = 0 \\
& & -T_3 & & +4T_5 & -T_6 & = 750 \\
& & & -T_4 & -2T_5 & +4T_6 & = 300
\end{array}$$

11.21 $T_a + T_b + T_d + T_e = 4T_c$

Example 42 *continued*

Writing this in matrix form gives

$$\begin{pmatrix} 4 & -1 & -1 & 0 & 0 & 0 \\ -2 & 4 & 0 & -1 & 0 & 0 \\ -1 & 0 & 4 & -1 & -1 & 0 \\ 0 & -1 & -2 & 4 & 0 & -1 \\ 0 & 0 & -1 & 0 & 4 & -1 \\ 0 & 0 & 0 & -1 & -2 & 4 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{pmatrix} = \begin{pmatrix} 900 \\ 450 \\ 450 \\ 0 \\ 750 \\ 300 \end{pmatrix}$$

Let **A** be the 6×6 matrix on the left-hand side, **t** be the temperature column vector and **b** be the column vector on the right-hand side. We have

$$\mathbf{At} = \mathbf{b}$$

Multiplying both sides by the inverse matrix, $\mathbf{A}^{-1}$, gives

$$\mathbf{t} = \mathbf{A}^{-1}\mathbf{b}$$

Using a graphical calculator:

Store the matrix **A** and the column vector **b** into a calculator and then find $\mathbf{A}^{-1}\mathbf{b}$.
The calculator should indicate the following results:

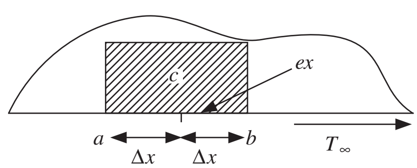
$T_1 = 439.29$, $T_2 = 435.27$, $T_3 = 421.88$, $T_4 = 412.5$, $T_5 = 385.71$ and $T_6 = 370.98$

(all temperatures are in Kelvin (K)).

For the procedure on a graphical calculator see the handbook of your calculator.

Normally to find the temperatures at these nodes we have to use a graphical calculator or a computer algebra system such as MAPLE, as shown above.

Fig. 26



The average temperature, T_{ex}, close to an exterior node *ex* (Fig. 26) is given by the formula

11.22

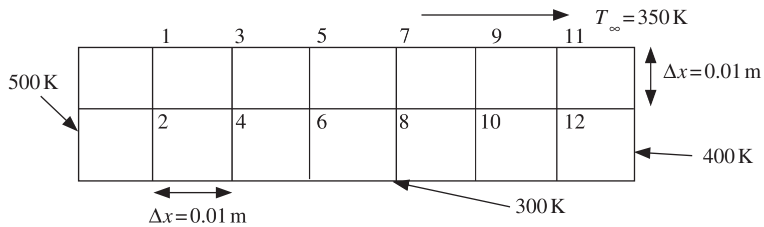
$$T_a + 2T_c + T_b + \frac{2h\Delta x}{k}(T_\infty) = 2\left(\frac{h\Delta x}{k} + 2\right)T_{ex}$$

where h is the heat transfer coefficient, k is the thermal conductivity and T_∞ is the exterior temperature.

For the interior node we are looking at conduction heat transfer only while for the exterior node we also consider the effects of convective heat transfer from the surrounding fluid.

Example 43

Fig. 27



Consider the rectangular bar of Fig. 27 with a thermal conductivity, $k = 1$ W/(m K) and heat transfer coefficient $h = 8$ W/(m^2 K). Find the temperatures at the nodes 1, 2, 3, 4, . . . , 11 and 12.

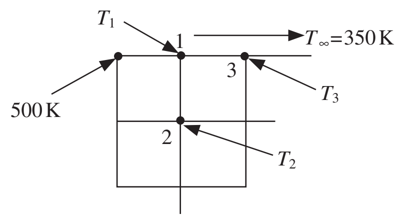
Solution

Let T_1, T_2, T_3, . . . , and T_{12} represent the temperature at the nodes 1, 2, 3, . . . , and 12 respectively.

The even nodes 2, 4, 6, . . . , 12 are interior nodes so we use 11.21 as in **Example 42.**

Node 2: $500 + T_1 + T_4 + 300 = 4T_2$
Node 4: $T_2 + T_3 + T_6 + 300 = 4T_4$
Node 6: $T_4 + T_5 + T_8 + 300 = 4T_6$
Node 8: $T_6 + T_7 + T_{10} + 300 = 4T_8$
Node 10: $T_8 + T_9 + T_{12} + 300 = 4T_{10}$
Node 12: $T_{10} + T_{11} + 400 + 300 = 4T_{12}$

Fig. 28



The odd nodes 1, 3, 5, . . . , 11 are exterior nodes so we use 11.22 by substituting $h = 8$, $\Delta x = 0.01$, $k = 1$, $T_\infty = 350$ and the given temperatures. Consider node 1 (Fig. 28).

Applying 11.22 with T_1 as exterior node gives

$$500 + 2T_2 + T_3 + \left(\frac{2 \times 8 \times 0.01}{1} \times 350\right) = 2\left[\left(\frac{8 \times 0.01}{1}\right) + 2\right]T_1$$

This simplifies the equation at node 1 to

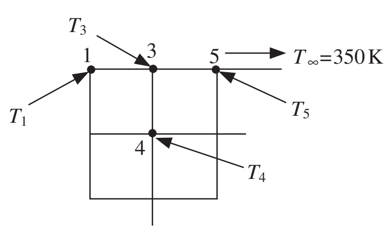
$$500 + 2T_2 + T_3 + 56 = 4.16T_1$$

Consider node 3 (Fig. 29).

Applying 11.22 to node 3 gives

$$T_1 + 2T_4 + T_5 + 56 = 4.16T_3$$

Fig. 29



11.21 $T_a + T_b + T_d + T_e = 4T_c$

11.22 $T_a + 2T_c + T_b + \dfrac{2h\Delta x}{k}(T_\infty) = 2\left(\dfrac{h\Delta x}{k} + 2\right)T_{ex}$

Example 43 *continued*

Similarly we have

Node 5: $T_3 + 2T_6 + T_7 + 56 = 4.16T_5$
Node 7: $T_5 + 2T_8 + T_9 + 56 = 4.16T_7$
Node 9: $T_7 + 2T_{10} + T_{11} + 56 = 4.16T_9$
Node 11: $T_9 + 2T_{12} + 400 + 56 = 4.16T_{11}$

Simplifying and rearranging these equations:

$$
\begin{aligned}
4.16T_1 - 2T_2 - T_3 &= 556 && \text{[Node 1]}\\
-T_1 + 4T_2 - T_4 &= 800 && \text{[Node 2]}\\
-T_1 + 4.16T_3 - 2T_4 - T_5 &= 56 && \text{[Node 3]}\\
-T_2 - T_3 + 4T_4 - T_6 &= 300 && \text{[Node 4]}\\
-T_3 + 4.16T_5 - 2T_6 - T_7 &= 56 && \text{[Node 5]}\\
-T_4 - T_5 + 4T_6 - T_8 &= 300 && \text{[Node 6]}\\
-T_5 + 4.16T_7 - 2T_8 - T_9 &= 56 && \text{[Node 7]}\\
-T_6 - T_7 + 4T_8 - T_{10} &= 300 && \text{[Node 8]}\\
-T_7 + 4.16T_9 - 2T_{10} - T_{11} &= 56 && \text{[Node 9]}\\
-T_8 - T_9 + 4T_{10} - T_{12} &= 300 && \text{[Node 10]}\\
-T_9 + 4.16T_{11} - 2T_{12} &= 456 && \text{[Node 11]}\\
-T_{10} - T_{11} + 4T_{12} &= 700 && \text{[Node 12]}
\end{aligned}
$$

In matrix form we have

$$
\begin{bmatrix}
4.16 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & 4.16 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & -1 & 4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 0 & 4.16 & -2 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & -1 & 4 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 0 & 4.16 & -2 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 4 & 0 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 4.16 & -2 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 4 & 0 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 4.16 & -2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 4
\end{bmatrix}
\begin{bmatrix} T_1\\ T_2\\ T_3\\ T_4\\ T_5\\ T_6\\ T_7\\ T_8\\ T_9\\ T_{10}\\ T_{11}\\ T_{12} \end{bmatrix}
=
\begin{bmatrix} 556\\ 800\\ 56\\ 300\\ 56\\ 300\\ 56\\ 300\\ 56\\ 300\\ 456\\ 700 \end{bmatrix}
$$

In the 12×12 matrix the first column represents the T_1 coefficients, the second column represents the T_2 coefficients, the third column represents the T_3 coefficients, and so on. **Why are there so many zeros in this matrix?**

Consider the first equation

$$4.16T_1 - 2T_2 - T_3 = 556$$

11.22 $$T_a + 2T_c + T_b + \frac{2h\Delta x}{k}(T_\infty) = 2\left(\frac{h\Delta x}{k} + 2\right)T_{ex}$$

Example 43 *continued*

Notice that there are no T_4, T_5, T_6, . . . , T_{12}. These are represented by the zeros in the first row, that is zeros in the T_4, T_5, T_6, . . . , T_{12} columns. Similarly we have zeros in all the other rows.

We need to use a computer algebra system (MAPLE) or a graphical calculator to find the temperatures at nodes 1, 2, 3, . . . , 11 and 12. **It is best to use MAPLE rather than a graphical calculator. Why?**

Because with a matrix of this size, 12 × 12, it is very easy to make an incorrect entry and with MAPLE you can clearly see the whole matrix with a reasonable size monitor. Also if you do make a mistake you can alter the incorrect entry. The MAPLE procedure is shown below, and it gives the following results:

$T_1 = 404.68$, $T_2 = 368.42$, $T_3 = 354.61$, $T_4 = 341.00$, $T_5 = 332.50$, $T_6 = 322.98$, $T_7 = 326.60$, $T_8 = 318.44$, $T_9 = 333.29$, $T_{10} = 324.17$, $T_{11} = 355.57$ and $T_{12} = 344.93$ (all temperatures are in Kelvin (K)).

```
> A:=matrix([[4.16,-2,-1, 0,0,0,0,0,0,0,0,0], [-1, 4, 0,
  -1,0,0,0,0,0,0,0,0], [-1, 0, 4.16,-2,-1,0,0,0,0,0,0,0], [0,-1,
  -1, 4,0,-1,0,0,0,0,0,0], [0, 0,-1,0,4.16,-2,-1,0,0,0,0,0],
  [0,0,0,-1,-1,4,0,-1,0,0,0,0],[0,0,0,0,-1,0,4.16,-2,-1,0,0,0],
  [0,0,0,0,0,-1,-1,4,0,-1,0,0],[0,0,0,0,0,0,-1,0,4.16,-2,-1,0],
  [0,0,0,0,0,0,0,1,-1,4,0,-1],[0,0,0,0,0,0,0,0,-1,0,4.16,-2],
  [0,0,0,0,0,0,0,0,0,-1,-1,4]]);
```

$$A:=\begin{bmatrix} 4.16 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 4.16 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 4.16 & -2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 4 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 4.16 & -2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 4 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 4.16 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 4.16 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 4 \end{bmatrix}$$

```
> b:=vector([556,800,56,300,56,300,56,300,56,300,456,700]);
```

$b := [556, 800, 56, 300, 56, 300, 56, 300, 56, 300, 456, 700]$

```
> linsolve(A,b);
```

404.6753628, 386.4197222, 354.6100646, 341.0035257, 332.4954545, 322.9843161, 326.6023941, 318.4382843, 333.2939364, 324.1664271, 355.5675273, 344.9334886]

```
> evalf(%,5);
```

[404.68, 386.42, 354.61, 341.00, 332.50, 322.98, 326.60, 318.44, 333.29, 324.17, 355.57, 344.93]

SUMMARY

The average interior temperature, T_c, close to a node c is given by

11.21 $$T_a + T_b + T_d + T_e = 4T_c$$

The average exterior temperature, T_{ex}, close to a node ex is given by

11.22 $$T_a + 2T_c + T_b + \frac{2h\Delta x}{k}(T_\infty) = 2\left(\frac{h\Delta x}{k} + 2\right)T_{ex}$$

Exercise **11(h)**

For all the questions in this exercise use a computer algebra system or a graphical calculator.

1

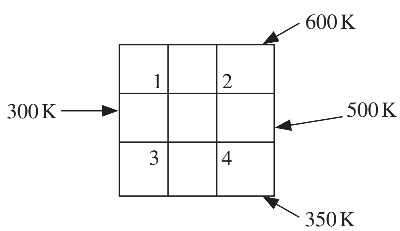
Fig.30


Figure 30 shows a square cross-section with the surface temperatures as indicated. Determine the temperatures at nodes 1, 2, 3 and 4. Do you think your results are correct?

2

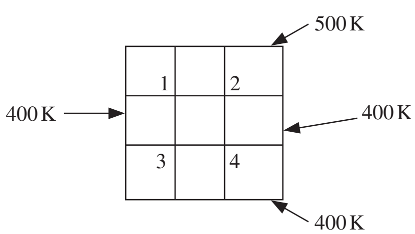
Fig.31


Figure 31 shows a square cross-section with the surface temperatures as indicated.

i Evaluate the temperatures at nodes 1, 2, 3 and 4.

ii What do you notice about your results?

iii Could you have found the temperatures of part **i** by using just two simultaneous equations?

3

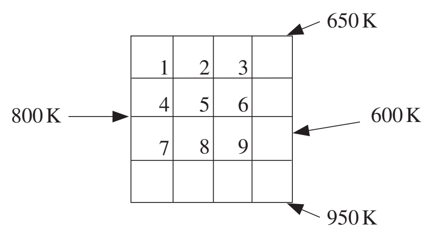
Fig.32


Figure 32 shows a square cross-section with the surface temperatures as indicated. Determine the temperatures at nodes 1, 2, . . . , 8 and 9.

4

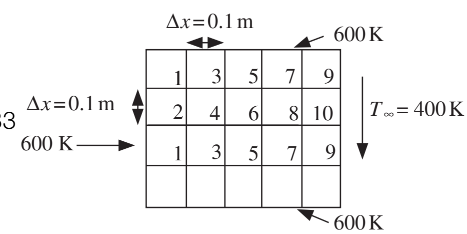
Fig.33


Figure 33 shows a rectangular cross-section with the surface temperatures as indicated. For $h = 10$ W/(m^2 K) and $k = 1$ W/(m K), find the temperatures at nodes 1, 2, 3, . . . , 9 and 10. Are there any other solutions?

Exercise 11(h) continued

5 Fig.34

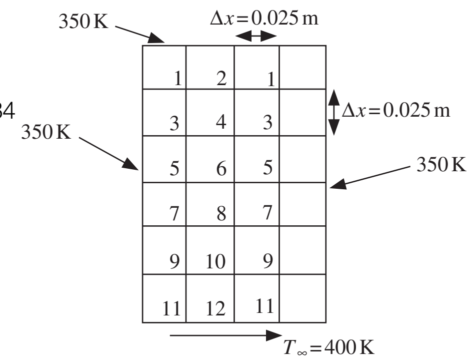


Figure 34 shows a rectangular cross-section of a bar. Determine the temperatures at nodes 1, 2, 3, . . . , 12 for $h = 50\ \text{W}/(\text{m}^2\,\text{K})$ and $k = 1\ \text{W}/(\text{m K})$.

Additional miscellaneous exercise 11

16 By using 11.6 and 11.7, find the controllability matrix, $\mathbf{C}_\text{T}$, and the observability matrix, $\mathbf{O}_\text{T}$, for the system with

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } \mathbf{C} = (3 \quad 4)$$

The system is said to be controllable if $\det \mathbf{C}_\text{T} \neq 0$ and is observable if $\det \mathbf{O}_\text{T} \neq 0$. Determine whether the above system is controllable and/or observable.

17 By using

$$\mathbf{C}_\text{T} = (\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}) \text{ and } \mathbf{O}_\text{T} = \begin{pmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{pmatrix}$$

find the controllability matrix, $\mathbf{C}_\text{T}$, and the observability matrix, $\mathbf{O}_\text{T}$, for the system with

$$\mathbf{A} = \begin{pmatrix} -1 & 2 & 5 \\ 7 & 3 & 1 \\ 6 & 8 & -11 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} \text{ and}$$

$$\mathbf{C} = (2 \quad 5 \quad 9)$$

18 A system has the following matrices:

$$\mathbf{A} = \begin{pmatrix} -3 & 4 \\ 1 & -5 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

i Determine the controllability matrix, $\mathbf{C}_\text{T} = (\mathbf{B} \quad \mathbf{AB})$.

ii Evaluate $\mathbf{C}_\text{T}^{-1}$.

iii We define $p(\mathbf{A})$ as

$p(\mathbf{A}) = \mathbf{A}^2 + k_1\mathbf{A} + k_0\mathbf{I}$ where $\mathbf{I}$ is the identity matrix

Determine $p(\mathbf{A})$ for $k_0 = 2$ and $k_1 = 2$.

iv The feedback matrix, $\mathbf{F}$, is determined by Ackermann's formula which is

$$\mathbf{F} = (0 \quad 1)\mathbf{C}_\text{T}^{-1}\, p(\mathbf{A})$$

Evaluate $\mathbf{F}$ for the above system.

19 The feedback matrix, $\mathbf{F}$, of a system can be evaluated by using Ackermann's formula

$$\mathbf{F} = (0 \quad 1)\mathbf{C}_\text{T}^{-1}\, p(\mathbf{A})$$

where $\mathbf{C}_\text{T}$ and $p(\mathbf{A})$ are as defined in question **18**. Determine $\mathbf{F}$ for

$$\mathbf{A} = \begin{pmatrix} -1 & 3 \\ 2 & 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

$k_0 = 3$ and $k_1 = 4$