CHAPTER 7

Engineering Applications of Differentiation

SECTION J Power series

By the end of this section you will be able to:

- test for convergence
- understand what is meant by a power series
- determine values of x of a power series for convergence

Here we examine the subject of power series. At first glance this section may seem rather abstract in comparison to previous chapters, and you may find the theoretical component of this topic initially hard to digest.

Don't be put off by the lack of practical, 'real-life' applications. Power series is widely used as a mathematical tool for understanding how electronic devices evaluate arguments of functions such as sin, cosine and logarithms, so it is worth taking the time to ensure you appreciate the mechanics of the following examples, even if the concepts seem less tangible to begin with.

Power series is important because functions such as exponential, sine and cosine can be defined as power series as demonstrated in the Maclaurin series section. These series contain a variable x and in this section we need to determine values of x for which the power series converges. To find these values of x we apply the ratio test which involves demanding algebraic skills.

J1 Test for Convergence

We can setup direct tests for convergence. A well-known series is the p-series which is:

7.30 **a** $\sum_{n=1}^{\infty}\left(\frac{1}{n^p}\right)$ converges for $p > 1$ (p is **greater than** 1)

7.30 **b** $\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n^p}\right)$ converges for $p \geq 1$ (p is **greater than or equal to** 1)

What does this (7.30)(a) mean?

7.30 **a** means that the infinite series $1/n^p$ converges if $p > 1$. **Does** $\Sigma\left(\frac{1}{n^3}\right)$ **converge?**

Yes because by (7.30) (a) we have $p = 3 > 1$ therefore the given series converges.

What does (7.30)(b) mean?

Writing out the terms of this series we have

$$\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n^p}\right) = \frac{(-1)^{1+1}}{1^p} + \frac{(-1)^{2+1}}{2^p} + \frac{(-1)^{3+1}}{3^p} + \frac{(-1)^{4+1}}{4^p} + \frac{(-1)^{5+1}}{5^p} + \cdots$$

$$= 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \frac{1}{5^p} - \cdots$$

This type of series is called an **alternating series** because it alternates between plus and minus terms. This series converges for $p = 1$ and $p > 1$. This means that the **alternating (harmonic) series**

$$\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n}\right) = \frac{(-1)^{1+1}}{1} + \frac{(-1)^{2+1}}{2} + \frac{(-1)^{3+1}}{3} + \frac{(-1)^{4+1}}{4} + \frac{(-1)^{5+1}}{5} + \cdots$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

converges because $p = 1$. [Note this series does converge at $p = 1$].

Next we state the ratio test for convergence which is very similar to the ratio test given in section H. See if you notice the difference between the two tests.

Ratio Test for Convergence 7.31.

Let $\sum(a_n)$ be a series of real numbers where $a_n \neq 0$ [Not zero] and

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \quad \text{[Modulus of ratio of the } (n+1)\text{th term to the } n\text{th term]}$$

i If $L < 1$ then the series $\sum(a_n)$ converges.
ii If $L > 1$ then the series $\sum(a_n)$ diverges.
iii If $L = 1$ the test fails and we **cannot** conclude whether the series converges or diverges.

What is the difference between this ratio test and 7.28 given in section H?

In the above case we find the ratio a_{n+1}/a_n and then take the **modulus** but in section H we did **not** take the modulus of the ratio a_{n+1}/a_n because we were dealing with positive terms and taking the modulus makes **no** difference to the ratio. However, in this section we may have negative terms a_n as seen in the alternating series above.

We can apply this ratio test for solving the next problem. You will need to understand your work on limits of functions of Chapter 3 because they are used to evaluate L. For example, we will use the following results of limits of functions:

$$\lim_{n\to\infty}\big(c\,f(n)\big) = c\lim_{n\to\infty}\big(f(n)\big) \text{ and } \lim_{n\to\infty}(c) = c \qquad \text{where } c \text{ is a constant}$$

We also need to use the following property of the modulus function:

$$|xy| = |x||y|$$

Example 38

Determine the values of x for which the following series converges $\sum\left(\frac{x^n}{n}\right)$.

Solution

To use the ratio test for convergence we need to find $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$. **What is a_n equal to?**

It is the nth term, so we have $a_n = \frac{x^n}{n}$. **How do we find a_{n+1}?**

a_{n+1} is the $(n+1)$th term so substituting $n+1$ for n into $a_n = \frac{x^n}{n}$ gives $a_{n+1} = \frac{x^{n+1}}{n+1}$.

Placing these, $a_n = \frac{x^n}{n}$ and $a_{n+1} = \frac{x^{n+1}}{n+1}$, into $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ we have

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}|a_{n+1} \div a_n| = \lim_{n\to\infty}\left|\left(\frac{x^{n+1}}{n+1}\right) \div \left(\frac{x^n}{n}\right)\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{x^{n+1}}{n+1}\right) \times \left(\frac{n}{x^n}\right)\right| \quad \left[\text{Inverting the second fraction and multiplying}\right]$$

$$= \lim_{n\to\infty}\left|x\left(\frac{n}{n+1}\right)\right| \quad \left[\text{Simplifying } \frac{x^{n+1}}{x^n} = \frac{x^n x}{x^n} = x\right]$$

$$= |x|\lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)\right| \quad \left[\text{Taking out } |x| \text{ because } \lim_{n\to\infty}(c\,f(n)) = c\lim_{n\to\infty}(f(n))\right]$$

How do we determine the limit in the last line, $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)$?

We divide the numerator and denominator by n which gives

$$L = |x|\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right) = |x|(1) = |x|\left[\text{Because as } n\to\infty \text{ therefore } \lim_{n\to\infty}\left(\frac{1}{1+1/n}\right) \to \frac{1}{1+0} = 1\right]$$

So far we have found $L = |x|$ but we need to answer the following question:

For what values of x does the series converge?

When $L < 1$ and we know $L = |x|$ so the given series converges for

$$L = |x| < 1 \quad \text{which means } -1 < x < 1$$

The test fails when $L = 1$, that is the values of x which satisfy $|x| = 1$ this means $x = 1$ or $x = -1$. **What happens at $x = 1$?**

Example 38 *continued*

Substituting $x = 1$ into the given series, $\sum\left(\frac{x^n}{n}\right)$, we have $\sum\left(\frac{1}{n}\right)$ which is the harmonic series and we know from section H, page 377 that this diverges. **What series do we have when $x = -1$?**

Substituting $x = -1$ into the given series, $\sum\left(\frac{x^n}{n}\right)$, we have $\sum\left(\frac{(-1)^n}{n}\right)$, which converges, demonstrated by the stated above p test (7.30) (b). Summarizing, the series converges for $-1 \le x < 1$:

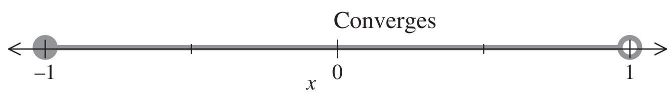

Fig. 42

J2 Power Series

A **power** series is a series of the form

$$c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots = \sum_{n=0}^{\infty}\left(c_nx^n\right)$$

where x is a real variable and $c_0, c_1, c_2, c_3, \cdots$ are constants. These constants are called the coefficients of the series. **Can you think of an example of a power series?**

Example 38 is a power series given by

$$\sum_{n=1}^{\infty}\left(\frac{x^n}{n}\right) = \underbrace{x}_{n=1} + \underbrace{\frac{x^2}{2}}_{n=2} + \underbrace{\frac{x^3}{3}}_{n=3} + \underbrace{\frac{x^4}{4}}_{n=4} + \cdots \qquad (*)$$

What are the values of the constants $c_0, c_1, c_2, c_3, \cdots$?

The coefficient c_0 is the constant coefficient, c_1 is the x coefficient, c_2 is the x^2 coefficient, c_3 is the x^3 coefficient etc. In the above case (*)

$$c_0 = 0 \text{ [Because there is no constant term]}, c_1 = 1, c_2 = \frac{1}{2}, c_3 = \frac{1}{3}, \cdots$$

The following are also examples of power series established in the Maclaurin series section on page 364:

(a) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right) = e^x$

(b) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = \sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n}}{(2n)!}\right) = \cos(x)$

(c) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right) = \sin(x)$

What are the values of the coefficients $c_0, c_1, c_2, c_3, c_4, c_5, \cdots$ in each of the above cases?
We have

(a) $c_0 = 1, c_1 = 1, c_2 = \frac{1}{2!}, c_3 = \frac{1}{3!}, c_4 = \frac{1}{4!}, c_5 = \frac{1}{5!}, \cdots$

(b) $c_0 = 1, c_1 = 0, c_2 = -\frac{1}{2!}, c_3 = 0, c_4 = \frac{1}{4!}, c_5 = 0, \cdots$

(c) $c_0 = 0, c_1 = 1, c_2 = 0, c_3 = -\frac{1}{3!}, c_4 = 0, c_5 = \frac{1}{5!}, \cdots$

Note that the coefficients, c_n, can be positive, negative or zero.

The above are examples of power series in x. Next we examine convergence of power series. Consider the general power series

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots = \sum_{n=0}^{\infty}\left(c_n x^n\right)$$

The series may **not** converge for all real numbers x. In the above Example 38 we found that the given series

$$\sum\left(\frac{x^n}{n}\right) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

converges for $-1 \le x < 1$. The interval $-1 \le x < 1$ is called the **interval of convergence** (see Fig. 42).

J3 Radius of Convergence

If a power series converges in the interval $-R < x < R$ then R is called the **radius of convergence**. The series may converge or diverge at $x = R$ or $x = -R$.

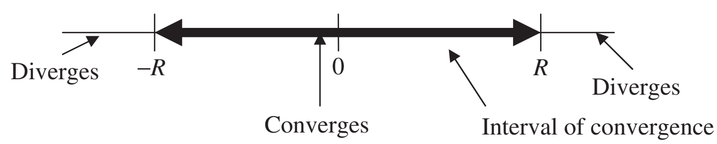

Fig. 43

The radius of convergence R for Example 38 is $R = 1$ because the interval of convergence is $-1 \le x < 1$.

For example, the power series for e^x is given by $\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)$ and has radius of convergence $R = +\infty$ and the interval of convergence is $-\infty < x < +\infty$. That is the series converges for all real numbers x as stated in formula (7.15) on page 364.

A different power series $\sum_{n=0}^{\infty}\left(n!\,x^n\right)$ has radius of convergence $R = 0$ and so it **only** converges at the **single** point $x = 0$.

J4 Testing Convergence of Power Series

We use the ratio test for convergence (7.31) to find the values of x for which the series converges.

Example 39

Determine the values of x for which the following power series converges $\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$.

Solution

How do we find the values of x?

Use the ratio test (7.31) where we need to find $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$.

What is a_n equal to in this example?

It is the nth term of the given power series, $a_n = \left(\frac{x}{2}\right)^n$. **What is a_{n+1} equal to in this example?**

Replacing n by $n + 1$ into $a_n = \left(\frac{x}{2}\right)^n$ gives

$$a_{n+1} = \left(\frac{x}{2}\right)^{n+1} \qquad [(n+1)\text{ th term}]$$

Substituting $a_n = \left(\frac{x}{2}\right)^n$ and $a_{n+1} = \left(\frac{x}{2}\right)^{n+1}$ into $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|a_{n+1} \div a_n\right|$ gives

$$L = \lim_{n\to\infty}\left|\left(\frac{x}{2}\right)^{n+1} \div \left(\frac{x}{2}\right)^n\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{x}{2}\right)^{n+1} \times \left(\frac{2}{x}\right)^n\right| \qquad \left[\text{Inverting the second fraction and multiplying}\right]$$

$$= \lim_{n\to\infty}\left|\frac{x^{n+1}}{2^{n+1}} \times \frac{2^n}{x^n}\right| \qquad \left[\text{Because } \left(\frac{x}{2}\right)^{n+1} = \frac{x^{n+1}}{2^{n+1}} \text{ and } \left(\frac{2}{x}\right)^n = \frac{2^n}{x^n}\right]$$

$$= \lim_{n\to\infty}\left|\frac{x^n x}{2^n 2} \times \frac{2^n}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{2}\right| \qquad \left[\text{Cancelling common factors } x^n \text{ and } 2^n\right]$$

$$L = \lim_{n\to\infty}\left|\frac{x}{2}\right| = \left|\frac{x}{2}\right| = \frac{|x|}{|2|} = \frac{|x|}{2} \qquad \left[\text{Because } \lim_{n\to\infty}(c) = c \text{ and } |2| = 2\right]$$

Example 39 *continued*

For what values of x does the series converge?

When $L = \frac{|x|}{2} < 1$ that is those x values which satisfy $\frac{|x|}{2} < 1$ therefore $|x| < 2$. Hence the given power series converges for $|x| < 2$ and diverges for $|x| > 2$. The values of x for convergence is $-2 < x < 2$. Remember the ratio test fails when $L = 1$ that is when $|x| = 2$ which means $x = 2$ or $x = -2$. **Does the series converge at $x = 2$ and $x = -2$?**

First substituting $x = 2$ into the given power series $\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$ we have

$$\sum_{n=0}^{\infty} \left(\frac{2}{2}\right)^n = \sum_{n=0}^{\infty} (1)^n$$

By

7.25 If $\lim_{n \to \infty} (a_n) \neq 0$ [Not zero] then $\Sigma(a_n)$ diverges

we conclude that this series $\sum_{n=0}^{\infty} (1)^n$ diverges because the nth term $(1)^n$ does **not** tend to zero, $\lim_{n \to \infty} (1)^n \neq 0$ [Not zero]. The given power series **diverges** for $x = 2$.

Similarly substituting $x = -2$ into the given power series $\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$ we have

$$\sum_{n=0}^{\infty} \left(\frac{(-2)}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n$$

Again by 7.25 the series diverges because $\lim_{n \to \infty} (-1)^n \neq 0$ [Not zero]. The given power series **diverges** for $x = -2$.

Summarizing, the interval of convergence is $-2 < x < 2$.

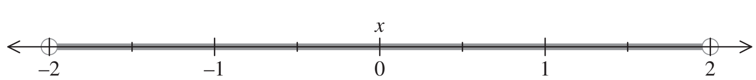

Fig. 44

What is the radius of convergence R in this case?

The radius of convergence, R, can be determined by $-R < x < R$. Since we have $-2 < x < 2$ therefore $R = 2$.

The next example involves a lot of algebraic simplification and application of limits of functions. You need to follow this example very carefully.

Example 40

Determine the interval and radius of convergence of the power series for $\sin(x)$ which is

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)$$

Solution

Again we apply the ratio test 7.31 with $L = \lim_{n\to 0}\left|\frac{a_{n+1}}{a_n}\right|$.

What is nth term, a_n, equal to in this case?

$$a_n = \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

What is (n+1)th term, a_{n+1}, equal to in this case?

Substituting $n+1$ for n into $a_n = \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ gives

$$a_{n+1} = \frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!} = \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!}$$

Putting these, $a_{n+1} = \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!}$ and $a_n = \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, into $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ gives

$$\begin{aligned}
L &= \lim_{n\to\infty}\left|\left(\frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!}\right) \div \left(\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)\right| \\
&= \lim_{n\to\infty}\left|\left(\frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!}\right) \times \left(\frac{(2n+1)!}{(-1)^n x^{2n+1}}\right)\right| && \left[\begin{array}{l}\text{Inverting the second}\\ \text{fraction and multiplying}\end{array}\right] \\
&= \lim_{n\to\infty}\left|\frac{(-1)^{n+1} x^{2n+3}}{(-1)^n x^{2n+1}} \times \left(\frac{(2n+1)!}{(2n+3)!}\right)\right| && [\text{collecting like terms}] \\
&= \lim_{n\to\infty}\left|(-1)x^2 \times \left(\frac{(2n+1)!}{(2n+3)!}\right)\right| && \left[\text{Because } \frac{(-1)^{n+1} x^{2n+3}}{(-1)^n x^{2n+1}} = \frac{(-1)^n(-1)x^{2n}x^3}{(-1)^n x^{2n} x} = (-1)x^2\right]
\end{aligned}$$

But how do we simplify the last line?

Using $|-1| = 1$ and $|x^2| = x^2$ because x^2 is positive (e.g., $1^2 = +1$, $(-1)^2 = +1$, $(-2)^2 = +4$ etc.) or zero ($0^2 = 0$) so $|(-1)\,x^2| = |-1||x^2| = x^2$ and

$$\lim_{n\to\infty}(cf(n)) = c\lim_{n\to\infty}(f(n))$$

Example 40 *continued*

On the last line we have

$$L = \lim_{n \to \infty} \left| (-1)\, x^2 \times \left(\frac{(2n+1)!}{(2n+3)!} \right) \right| = x^2 \left(\lim_{n \to \infty} \frac{(2n+1)!}{(2n+3)!} \right) \qquad (\dagger)$$

How do we simplify the large bracket term in (†)?

Well

$$(2n+1)! = 1 \times 2 \times 3 \times \ldots \times 2n \times (2n+1)$$

$$(2n+3)! = 1 \times 2 \times 3 \times \ldots \times 2n \times (2n+1) \times (2n+2) \times (2n+3)$$

$$\frac{(2n+1)!}{(2n+3)!} = \frac{1 \times 2 \times 3 \times \ldots \times 2n \times (2n+1)}{1 \times 2 \times 3 \times \ldots \times 2n \times (2n+1) \times (2n+2) \times (2n+3)}$$

$$= \frac{1}{(2n+2) \times (2n+3)} \qquad \left[\begin{array}{l} \text{Cancelling ALL common} \\ \text{factors } 1, 2, 3, \cdots, 2n \text{ and } 2n+1 \end{array} \right]$$

Evaluating this limit by expanding the brackets in the denominator gives

$$\lim_{n \to \infty} \left(\frac{1}{(2n+3)\,(2n+1)} \right) = \lim_{n \to \infty} \left(\frac{1}{4n^2 + 8n + 3} \right) = 0 \left[\text{Because as n} \to \infty \text{ then } \frac{1}{4n^2 + 8n + 3} \to 0 \right]$$

Substituting this, $\lim_{n \to \infty} \left(\frac{(2n+1)!}{(2n+3)!} \right) = 0$, into (†) gives

$$L = x^2(0) = 0$$

For what values of x does the series converge?

For all real numbers x because for **all** x we have $L = 0 < 1$. The interval of convergence is $-\infty < x < +\infty$. The radius of convergence is $R = +\infty$. This was noted in 7.16 on page 364, where we stated the sin series expansion was valid for **all** x which means $-\infty < x < +\infty$.

Until now we have considered a power series about the point $x = 0$. A more general power series is of the form

$$\sum_{n=0}^{\infty} c_n (x - b)^n$$

where b is a real fixed number. We say the power series is expanded about the point b. When $b = 0$ then the power series is expanded about 0 as in previous examples.

$$\sum_{n=0}^{\infty} \left(c_n x^n \right) \quad [\text{Power series about the point } b = 0]$$

How do you write a power series expanded about the point $b = 1$?

$$\sum_{n=0}^{\infty} c_n \left(x - 1\right)^n$$

How do you write a power series expanded about the point $b = -2$?

$$\sum_{n=0}^{\infty} c_n \left[x - (-2)\right]^n = \sum_{n=0}^{\infty} c_n (x + 2)^n$$

Example 41

Determine the interval and radius of convergence of the power series

$$\sum_{n=1}^{\infty} \left(\frac{(x + 1)^n}{3^n}\right)$$

Solution

Note that the given power series is expanded about the point $b = -1$.

We apply the ratio test 7.31 with $L = \lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right|$ to find the interval of convergence.

What is a_n equal to in this case?

$$a_n = \frac{(x + 1)^n}{3^n}$$

What is a_{n+1} equal to?

Substituting $n + 1$ for n into $a_n = \frac{(x + 1)^n}{3^n}$ gives

$$a_{n+1} = \frac{(x + 1)^{n+1}}{3^{n+1}}$$

Putting $a_n = \frac{(x + 1)^n}{3^n}$ and $a_{n+1} = \frac{(x + 1)^{n+1}}{3^{n+1}}$ into $L = \lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right|$ gives

$$L = \lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty} \left|\frac{(x + 1)^{n+1}}{3^{n+1}} \times \frac{3^n}{(x + 1)^n}\right|$$

$$= \lim_{n \to \infty} \left|\frac{(x + 1)^n (x + 1)}{3^n 3} \times \frac{3^n}{(x + 1)^n}\right| \quad \left[\begin{array}{l}\text{Writing } 3^{n+1} = 3^n 3 \text{ and} \\ (x + 1)^{n+1} = (x + 1)^n (x + 1)\end{array}\right]$$

$$= \lim_{n \to \infty} \left|\frac{x + 1}{3}\right| = \left|\frac{x + 1}{3}\right| \quad \left[\begin{array}{l}\text{Cancelling out } 3^n \text{ and} \\ (x + 1)^n\end{array}\right]$$

The series converges for $L = \frac{|x + 1|}{3} < 1$ which gives $|x + 1| < 3$. **What values of x satisfy $|x + 1| < 3$?**

Example 41 *continued*

This means that x lies at the centre -1 with distance 3 away from -1.

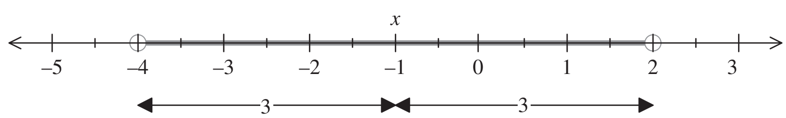

Fig. 45

Hence the given power series converges for $-4 < x < 2$. Outside this range the series diverges that is if $x > 2$ and $x < -4$ the given power series diverges.

The test fails for $L = 1$ that is $|x + 1| = 3$ [Equal to 3] which means $x = 2$ or $x = -4$.

Does the series converge when $x = 2$ and $x = -4$?

Substituting $x = 2$ into the given power series $\sum_{n=1}^{\infty}\left(\frac{(x+1)^n}{3^n}\right)$ we have

$$\sum_{n=1}^{\infty}\left(\frac{(2+1)^n}{3^n}\right) = \sum_{n=1}^{\infty}\left(\frac{3^n}{3^n}\right) = \sum_{n=1}^{\infty}(1) \quad \text{[Cancelling } 3^n \text{ s]}$$

By

7.25 If $\lim_{n \to \infty}(a^n) \neq 0$ [not zero] then $\Sigma\,(a_n)$ diverges

we conclude that this series $\sum_{n=1}^{\infty}(1)$ diverges because the nth term, 1, does **not** tend to zero.

Similarly substituting the other value $x = -4$ into the given power series $\sum_{n=1}^{\infty}\left(\frac{(x+1)^n}{3^n}\right)$ we have

$$\sum_{n=1}^{\infty}\left(\frac{(-4+1)^n}{3^n}\right) = \sum_{n=1}^{\infty}\left(\frac{(-3)^n}{3^n}\right) = \sum_{n=1}^{\infty}\left(\frac{-3}{3}\right)^n = \sum_{n=1}^{\infty}\left(-1\right)^n$$

Similarly by (7.25) this series $\sum_{n=1}^{x}(-1)^n$ diverges because the nth term $(-1)^n$ does **not** tend to zero as $n \to \infty$.

Hence the interval of convergence is $-4 < x < 2$.

What is the radius of convergence, R, equal to in this case?

Radius of convergence $R = 3$. (See Fig. 45).

Determining values of x for which the power series converges is not a difficult task but it does involve a number of different topics. You need to thoroughly understand evaluation of limits of functions, algebraic simplification, inequalities and properties of the modulus function.

SUMMARY

A general power series is of the form

$$c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots = \sum_{n=0}^{\infty}\left(c_nx^n\right)$$

We can find the interval of convergence for which the power series converges by using the following ratio test:
Ratio Test for Convergence (7.31).

Let $\sum\left(a_n\right)$ be a series of real numbers where $a_n \neq 0$ [Not zero] for all n and

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$$

I If $L < 1$ then the series $\sum(a_n)$ converges.
II If $L > 1$ then the series $\sum(a_n)$ diverges.
III If $L = 1$ we **cannot** conclude whether the series converges or diverges.

Exercise 7(j)

1 Determine the interval and radius of convergence of the following power series:

a $\sum_{n=0}^{\infty}(x^n)$ **b** $\sum_{n=1}^{\infty}\left(\frac{x^n}{2n}\right)$

c $\sum_{n=1}^{\infty}\left(\frac{x^n}{n^2}\right)$ **d** $\sum_{n=1}^{\infty}\left(\frac{nx^n}{2n+1}\right)$

e $\sum_{n=0}^{\infty}\left(\frac{x}{\sqrt{2}}\right)$ **f** $\sum_{n=0}^{\infty}\left(\frac{n^2}{2^n}\right)x^n$

2 Determine the interval of convergence of:

a $\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)$ **b** $\sum_{n=1}^{\infty}\left(\frac{(-1)^{n-1}x^n}{n}\right)$

c $\sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n}}{(2n)!}\right)$ **d** $\sum_{n=0}^{\infty}\left(\frac{(-1)^n x^{2n+1}}{(2n+1)}\right)$

e $\sum_{n=0}^{\infty}\left(x^{2n}\right)$ **f** $\sum_{n=0}^{\infty}\left(x^{n^2}\right)$

3 Determine the interval of convergence of:

a $\sum_{n=1}^{\infty}\left(\frac{(x+1)^n}{n}\right)$ **b** $\sum_{n=1}^{\infty}\left(\frac{(x-2)^n}{n^2}\right)$

c $\sum_{n=0}^{\infty}\left(\frac{(-1)^n(x-1)^n}{2^n}\right)$

d $\sum_{n=0}^{\infty}\left(\frac{n}{2n+1}\left(\frac{x+3}{2}\right)^n\right)$

4 Show that $L = 1$ for each of the following series:

a $\sum\left(\frac{1}{n}\right)$ **b** $\sum\left(\frac{(-1)^{n+1}}{n^2}\right)$

c $\sum\left(\frac{(-1)^{n+1}}{n}\right)$

5 Find the interval of convergence of:

a $\sum_{n=0}^{\infty}\left(nx^{n-1}\right)$ **b** $\sum_{n=1}^{\infty}\left(\frac{(-1)^n x^n}{\sqrt{n}}\right)$

c $\sum_{n=0}^{\infty}\left(n^p x^n\right)$ **d** $\sum_{n=1}^{\infty}\left(\frac{x^n}{n^2\sqrt{n}}\right)$

e $\sum_{n=1}^{\infty}\left(\frac{(-3)^n x^n}{n\sqrt{n}}\right)$ **f** $\sum_{n=0}^{\infty}\left(x^n n^2 e^{-2n}\right)$

Solutions

1 **a** Interval of convergence is $-1 < x < 1$. Radius of convergence is $R = 1$.
b Interval of convergence is $-1 \le x < 1$. Radius of convergence is $R = 1$.
c Interval of convergence is $-1 \le x \le 1$. Radius of convergence is $R = 1$.
d Interval of convergence is $-1 < x < 1$. Radius of convergence is $R = 1$.
e Interval of convergence is $-\sqrt{2} < x < \sqrt{2}$. Radius of convergence is $R = \sqrt{2}$.
f Interval of convergence is $-2 < x < 2$. Radius of convergence is $R = 2$.

2 Interval of convergence is:

a $-\infty < x < +\infty$ **b** $-1 < x \le 1$
c $-\infty < x < +\infty$ **d** $-1 \le x \le 1$
e $-1 < x < 1$ **f** $-1 < x < 1$

3 Interval of convergence is

a $-2 \le x < 0$ **b** $1 \le x \le 3$
c $-1 < x < 3$ **d** $-5 < x < -1$

4 **a** $\lim_{n \to \infty}\left(\frac{n}{n+1}\right) = 1$ **b** $\lim_{n \to \infty}\left(\frac{n}{n+1}\right)^2 = 1$ **c** $\lim_{n \to \infty}\left(\frac{n}{n+1}\right) = 1$

5 Interval of convergence is

a $-1 < x < 1$ **b** $-1 < x \le 1$ **c** $-1 < x < 1$
d $-1 \le x \le 1$ **e** $-\frac{1}{3} \le x \le \frac{1}{3}$ **f** $-e^2 < x < e^2$