Exercise 5e

15.(i) From $A + B = \theta_0$ we have $B = \theta_0 - A$. Substituting this into $\theta_L = Ae^{kL} + Be^{-kL}$ gives:

$$\begin{aligned}\theta_L &= Ae^{kL} + (\theta_0 - A)e^{-kL} \\ &= A(e^{kL} - e^{-kL}) + \theta_0 e^{-kL}\end{aligned}$$

Rearranging: $$A(e^{kL} - e^{-kL}) = \theta_L - \theta_0 e^{-kL}$$

$$A = \frac{\theta_L - \theta_0 e^{-kL}}{e^{kL} - e^{-kL}}$$

(ii) $$B = \theta_0 - A \underbrace{=}_{\text{from part (i)}} \theta_0 - \frac{\theta_L - \theta_0 e^{-kL}}{e^{kL} - e^{-kL}}$$

$$= \frac{\theta_0\left(e^{kL} - e^{-kL}\right) - \theta_L + \theta_0 e^{-kL}}{e^{kL} - e^{-kL}}$$

$$= \frac{\theta_0 e^{kL} - \theta_0 e^{-kL} - \theta_L + \theta_0 e^{-kL}}{e^{kL} - e^{-kL}}$$

$$B = \frac{\theta_0 e^{kL} - \theta_L}{e^{kL} - e^{-kL}}$$

(iii) Substituting for A and B from parts (i) and (ii) gives:

$$\theta(x) = Ae^{kx} + Be^{-kx} = \frac{\left(\theta_L - \theta_0 e^{-kL}\right)e^{kx} + \left(\theta_0 e^{kL} - \theta_L\right)e^{-kx}}{e^{kL} - e^{-kL}}$$

$$= \frac{\theta_L e^{kx} - \theta_0 e^{-kL+kx} + \theta_0 e^{kL-kx} - \theta_L e^{-kx}}{e^{kL} - e^{-kL}}$$

$$= \frac{\theta_L\left(e^{kx} - e^{-kx}\right) + \theta_0\left[e^{k(L-x)} - e^{-k(L-x)}\right]}{e^{kL} - e^{-kL}}$$

$$\theta(x) \underbrace{=}_{\text{by (5.24)}} \frac{\theta_L \sinh(kx) + \theta_0 \sinh\left[k(L-x)\right]}{\sinh(kL)}$$

(5.24) $$\left(e^{x} - e^{-x}\right)/2 = \sinh(x)$$