16. Multiplying the matrices gives

$$\mathbf{AB} = \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

Using $\mathbf{C}_T = (\mathbf{B} \quad \mathbf{AB})$ we have

$$\mathbf{C}_T = \begin{pmatrix} 2 & 7 \\ 1 & 3 \end{pmatrix} \text{ and } \det \mathbf{C}_T = (2\times 3) - (1\times 7) = -1 \neq 0$$

Hence the system is controllable. Next we find $\mathbf{O}_T = \begin{pmatrix} \mathbf{C} \\ \mathbf{CA} \end{pmatrix}$

$$\mathbf{CA} = (3 \quad 4)\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix} = (2 \quad 29)$$

$$\mathbf{O}_T = \begin{pmatrix} 3 & 4 \\ 2 & 29 \end{pmatrix} \text{ and } \det \mathbf{O}_T = (3\times 29) - (2\times 4) \neq 0$$

The system is also observable.

17. Multiplying the matrices for $\mathbf{C}_T = (\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B})$ gives

$$\mathbf{AB} = \begin{pmatrix} -1 & 2 & 5 \\ 7 & 3 & 1 \\ 6 & 8 & -11 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 44 \\ 29 \\ -31 \end{pmatrix}$$

Also

$$\mathbf{A}^2\mathbf{B} = \mathbf{A}(\mathbf{AB}) = \begin{pmatrix} -1 & 2 & 5 \\ 7 & 3 & 1 \\ 6 & 8 & -11 \end{pmatrix}\begin{pmatrix} 44 \\ 29 \\ -31 \end{pmatrix} = \begin{pmatrix} -141 \\ 364 \\ 837 \end{pmatrix}$$

Substituting these into $\mathbf{C}_T = (\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B})$ gives

$$\mathbf{C}_T = \begin{pmatrix} 1 & 44 & -141 \\ 5 & 29 & 364 \\ 7 & -31 & 837 \end{pmatrix}$$

Similarly for $\mathbf{O}_T$ we have

$$\mathbf{CA} = (2 \quad 5 \quad 9)\begin{pmatrix} -1 & 2 & 5 \\ 7 & 3 & 1 \\ 6 & 8 & -11 \end{pmatrix} = (87 \quad 91 \quad -84)$$

$$\mathbf{CA}^2 = (\mathbf{CA})\mathbf{A} = (87 \quad 91 \quad -84)\begin{pmatrix} -1 & 2 & 5 \\ 7 & 3 & 1 \\ 6 & 8 & -11 \end{pmatrix} = (46 \quad -225 \quad 1450)$$

$$\mathbf{O}_T = \begin{pmatrix} 2 & 5 & 9 \\ 87 & 91 & -84 \\ 46 & -225 & 1450 \end{pmatrix}$$

18. (i) We use $\mathbf{C}_T = (\mathbf{B} \quad \mathbf{AB})$

$$\mathbf{AB} = \begin{pmatrix} -3 & 4 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -7 \\ 6 \end{pmatrix}$$

Thus $\mathbf{C}_T = \begin{pmatrix} 1 & -7 \\ -1 & 6 \end{pmatrix}$

(ii) By (11.4)

$$\mathbf{C}_T^{-1} = \begin{pmatrix} 1 & -7 \\ -1 & 6 \end{pmatrix}^{-1} = \frac{1}{6-7}\begin{pmatrix} 6 & 7 \\ 1 & 1 \end{pmatrix} = -1\begin{pmatrix} 6 & 7 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -6 & -7 \\ -1 & -1 \end{pmatrix}$$

(iii) By substituting $k_0 = 2$ and $k_1 = 2$ we have $p(\mathbf{A}) = \mathbf{A}^2 + 2\mathbf{A} + 2\mathbf{I}$.

$$\mathbf{A}^2 = \begin{pmatrix} -3 & 4 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} -3 & 4 \\ 1 & -5 \end{pmatrix} = \begin{pmatrix} 13 & -32 \\ -8 & 29 \end{pmatrix}$$

Hence

$$p(\mathbf{A}) = \begin{pmatrix} 13 & -32 \\ -8 & 29 \end{pmatrix} + 2\begin{pmatrix} -3 & 4 \\ 1 & -5 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & -32 \\ -8 & 29 \end{pmatrix} + \begin{pmatrix} -6 & 8 \\ 2 & -10 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 13-6+2 & -32+8+0 \\ -8+2+0 & 29-10+2 \end{pmatrix} = \begin{pmatrix} 9 & -24 \\ -6 & 21 \end{pmatrix}$$

(iv) Substituting these into $\mathbf{F}$ gives

$$\mathbf{F} = (0 \quad 1)\begin{pmatrix} -6 & -7 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 9 & -24 \\ -6 & 21 \end{pmatrix} = (0 \quad 1)\begin{pmatrix} -12 & -3 \\ -3 & 3 \end{pmatrix} = (-3 \quad 3)$$

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19. Similar to solution 18. We have

$$\mathbf{C}_T = \begin{pmatrix} 1 & 5 \\ 2 & 12 \end{pmatrix}$$

By (11.4)

$$\mathbf{C}_T^{-1} = \begin{pmatrix} 1 & 5 \\ 2 & 12 \end{pmatrix}^{-1} = \frac{1}{12-10}\begin{pmatrix} 12 & -5 \\ -2 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 12 & -5 \\ -2 & 1 \end{pmatrix}$$

By substituting $k_0 = 3$ and $k_1 = 4$ into $p(\mathbf{A})$

$$p(\mathbf{A}) = \mathbf{A}^2 + 4\mathbf{A} + 3\mathbf{I} \qquad (*)$$

$$\mathbf{A}^2 = \begin{pmatrix} -1 & 3 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 7 & 12 \\ 8 & 31 \end{pmatrix}$$

Substituting into (*) gives

$$p(\mathbf{A}) = \begin{pmatrix} 7 & 12 \\ 8 & 31 \end{pmatrix} + 4\begin{pmatrix} -1 & 3 \\ 2 & 5 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 7-4+3 & 12+12+0 \\ 8+8+0 & 31+20+3 \end{pmatrix} = \begin{pmatrix} 6 & 24 \\ 16 & 54 \end{pmatrix}$$

Thus the feedback matrix, **F**, is given by

$$\mathbf{F} = (0 \quad 1)\frac{1}{2}\begin{pmatrix} 12 & -5 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 6 & 24 \\ 16 & 54 \end{pmatrix} = \frac{1}{2}(0 \quad 1)\begin{pmatrix} -8 & 18 \\ 4 & 6 \end{pmatrix} = \frac{1}{2}(4 \quad 6) = (2 \quad 3)$$

(11.4) $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-cb}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$