Chapter 10

SECTION F **Functions of complex numbers**

By the end of this section you will be able to:

- use some identities between trigonometric and hyperbolic functions
- establish some of these identities
- apply these to engineering examples

This section is a lot **more** difficult than previous sections. In this section we establish and state a number of identities involving complex numbers.

F1 **Identities**

In this section we use the fundamental identities derived in the previous section.

10.25 $$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

10.26 $$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

Remember that θ needs to be in radians.

Example 29

Show that

$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \cos(\theta)$$

Solution

Expanding the Left-Hand Side:

$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \frac{\overbrace{\cos(\theta) + j\sin(\theta)}^{\text{by } 10.25} + \overbrace{\cos(\theta) - j\sin(\theta)}^{\text{by } 10.26}}{2}$$

$$= \frac{2\cos(\theta)}{2} = \cos(\theta) \quad \text{[Cancelling 2's]}$$

We define the complex trigonometric functions $\sin(z)$ and $\cos(z)$ as follows:

10.27 $$\cos(z) = \frac{e^{jz} + e^{-jz}}{2} \quad \text{[Replace } \theta \text{ by } z \text{ in } \textbf{Example 29}\text{]}$$

Similarly

10.28 $$\sin(z) = \frac{e^{jz} - e^{-jz}}{2j}$$

From these we can obtain

10.29 $$\tan(z) = \frac{\sin(z)}{\cos(z)} = \frac{1}{j}\left[\frac{e^{jz} - e^{-jz}}{e^{jz} + e^{-jz}}\right]$$

We define the complex hyperbolic functions as

10.30 $$\cosh(z) = \frac{e^{z} + e^{-z}}{2}$$

10.31 $$\sinh(z) = \frac{e^{z} - e^{-z}}{2}$$

10.32 $$\tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$$

Example 30

Show that

$$\sin(jz) = j\sinh(z)$$

Solution

By

10.28 $$\sin(z) = (e^{jz} - e^{-jz})/2j$$

$$\sin(jz) = \frac{e^{j(jz)} - e^{-j(jz)}}{2j}$$

$$= \frac{e^{j^2 z} - e^{-j^2 z}}{2j}$$

$$= \frac{e^{-z} - e^{z}}{2j} \quad \text{[Because } j^2 = -1\text{]}$$

$$\underbrace{=}_{\text{by } 10.13} \frac{-j2(e^{-z} - e^{z})}{4} \quad \text{[Complex conjugate of } j2 \text{ is } -j2\text{]}$$

$$= -j\left(\frac{e^{-z} - e^{z}}{2}\right) \quad \left[\text{Because } \frac{2}{4} = \frac{1}{2}\right]$$

$$= j\left(\frac{e^{z} - e^{-z}}{2}\right) = j\underbrace{\sinh(z)}_{\text{by } 10.31}$$

10.13 $\frac{a + jb}{c + jd} = \frac{(a + jb)(c - jd)}{c^2 + d^2}$ 10.31 $\sinh(z) = (e^{z} - e^{-z})/2$

Similarly we have the following identities:

10.33 $\cos(jz) = \cosh(z)$

10.34 $\sin(jz) = j\sinh(z)$

10.35 $\tan(jz) = j\tanh(z)$

10.36 $\cosh(jz) = \cos(z)$

10.37 $\sinh(jz) = j\sin(z)$

10.38 $\tanh(jz) = j\tan(z)$

You are asked to show some of these identities in **Exercise 10(f)**. Many of the properties of real trigonometric functions also apply to complex trigonometric functions. We will not list them here but just apply them in the appropriate case, as the following example shows.

Example 31

Determine x and y given that

$$x + jy = \cos(0.12 + j3)$$

(x and y are real.)

Solution

We use

4.39 $\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2)$

$$\cos(0.12 + j3) = \cos(0.12)\cos(j3) - \sin(0.12)\sin(j3)$$

* $$= 0.99\cos(j3) - 0.12\sin(j3)$$

? **What is cos(j3) and sin(j3) equal to?**

$$\cos(j3) \underbrace{=}_{\text{by } 10.33} \cosh(3) = 10.07 \quad \text{[Via calculator]}$$

$$\sin(j3) \underbrace{=}_{\text{by } 10.34} j\sinh(3) = j10.02 \quad \text{[Via calculator]}$$

Substituting these values into * gives

$$\cos(0.12 + j3) = (0.99 \times 10.07) - (0.12 \times j10.02)$$

$$= 9.97 - j1.20$$

Equating the real and imaginary parts of $x + jy = 9.97 - j1.20$ gives

$$x = 9.97 \text{ and } y = -1.20$$

10.33 $\cos(jz) = \cosh(z)$ 10.34 $\sin(jz) = j\sinh(z)$

The next example might seem like a colossal jump from previous examples. Don't be put off by all the different symbols used in the example, we still use the same rules of complex numbers.

Example 32 *electrical principles*

A transmission line of length L with a characteristic impedance z_0 has an input impedance z_{input} given by

$$z_{\text{input}} = z_0\left[\frac{z_L\cosh(\gamma L) + z_0\sinh(\gamma L)}{z_0\cosh(\gamma L) + z_L\sinh(\gamma L)}\right] \qquad \dagger$$

where γ = propagation coefficient and z_L = load. Show that if $\gamma = j\beta$ then

$$z_{\text{input}} = z_0\left[\frac{z_L + jz_0\tan(\beta L)}{z_0 + jz_L\tan(\beta L)}\right]$$

Solution

Dividing the numerator and denominator of † by $\cosh(\gamma L)$:

$$z_{\text{input}} = z_0\left[\frac{z_L + z_0\dfrac{\sinh(\gamma L)}{\cosh(\gamma L)}}{z_0 + z_L\dfrac{\sinh(\gamma L)}{\cosh(\gamma L)}}\right] \qquad \left[\text{Remember } \frac{\cosh(\gamma L)}{\cosh(\gamma L)} = 1\right]$$

$$\underbrace{=}_{\text{by } 5.27} z_0\left[\frac{z_L + z_0\tanh(\gamma L)}{z_0 + z_L\tanh(\gamma L)}\right]$$

$$= z_0\left[\frac{z_L + z_0\tanh(j\beta L)}{z_0 + z_L\tanh(j\beta L)}\right] \qquad [\text{Substituting } \gamma = j\beta]$$

$$\underbrace{=}_{\text{by } 10.38} z_0\left[\frac{z_L + jz_0\tan(\beta L)}{z_0 + jz_L\tan(\beta L)}\right]$$

SUMMARY

There are many identities showing relationships between hyperbolic and trigonometric functions. We can use these to evaluate trigonometric and hyperbolic functions of complex numbers.

5.27 $\dfrac{\sinh(x)}{\cosh(x)} = \tanh(x)$ 10.38 $\tanh(jz) = j\tan(z)$

Exercise **10(f)**

Solutions are given at the end of this additional material. Complete solutions are in this website.

1 Evaluate the following:

a $\cos(j)$

b $\sin(j)$

c $\tan(j)$

d $\sin(j\pi)$

2 Evaluate the following:

a $\cosh(j\pi)$

b $\sinh(j\ln(3))$

c $\tanh(j\pi/3)$

3 Show that

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

4 Show that

$$\tan(z) = \frac{1}{j}\left[\frac{e^{jz} - e^{-jz}}{e^{jz} + e^{-jz}}\right]$$

5 By using 10.27 and 10.28 show that

$$\cos^2(z) + \sin^2(z) = 1$$

6 Show that

$$\cos(jz) = \cosh(z)$$

7 Show that

$$\cosh(jz) = \cos(z)$$

8 Determine values of x and y for each of the following (x and y are real):

a $x + jy = \cos(1 + j\pi)$

b $x + jy = \sin(-1 - j\pi)$

c $x + jy = \tanh\left(-j\frac{\pi}{4}\right)$

9 [*electrical principles*] A cable has a voltage v at a distance x from the sending end, given by

$$v = V_L\left(\cosh(\gamma x) + \frac{z_0}{z_L}\sinh(\gamma x)\right)$$

where V_L is the load voltage, z_0 is the characteristic impedance, z_L is the load impedance and γ is the propagation coefficient. Show that if $\gamma = j\beta$ then

$$v = V_L\left(\cos(\beta x) + j\frac{z_0}{z_L}\sin(\beta x)\right)$$

10 [*electrical principles*] If a voltage v at a distance x along a cable is given by

$$v = I_L z_0 \sinh[(\gamma + j\beta)x]$$

show that

$$\frac{v}{I_L z_0} = \sinh(\gamma x)\cos(\beta x) + j\cosh(\gamma x)\sin(\beta x)$$

11 [*electrical principles*] The impedance, z_x, at a distance x along a transmission line is given by

$$z_x = z_0\frac{(z_0 + z_L)e^{\gamma x} + (z_L - z_0)e^{-\gamma x}}{(z_0 + z_L)e^{\gamma x} + (z_0 - z_L)e^{-\gamma x}}$$

where z_L is the load impedance, z_0 is the characteristic impedance and γ is the propagation coefficient. Show that

$$z_x = z_0\left[\frac{z_0\sinh(\gamma x) + z_L\cosh(\gamma x)}{z_0\cosh(\gamma x) + z_L\sinh(\gamma x)}\right]$$

Miscellaneous exercise **10 (extra)**

Solutions are given at the end of this additional material. Complete solutions are in this website.

21 Evaluate the following:

a $\cos(\ln(1) + j)$ **b** $\sin\left(\frac{\pi}{2} + j\right)$

22 [*electrical principles*]

i The current, I_x, in a transmission line at a distance x from the receiving end is given by

$$I_x = \frac{I_L}{2z_0}\left(e^{\gamma x}(z_0 + z_L) - (z_0 - z_L)e^{-\gamma x}\right)$$

where I_L is the load current, z_0 is the characteristic impedance, z_L is the load impedance and γ is the propagation coefficient. Show that

$$I_x = I_L\left(\sinh(\gamma x) + \frac{z_L}{z_0}\cosh(\gamma x)\right)$$

ii Evaluate I_x for $\gamma x = 0.01 + j0.1$, $z_L = 250\angle 10°\ \Omega$, $z_0 = 500\angle(-10°)\ \Omega$ and $I_L = 250\text{A}$.

23 [*control engineering*] The steady-state output, y_{ss}, of a stable system is given by

$$y_{ss} = C\left[\frac{e^{j\phi}e^{j\omega t} - e^{-j\phi}e^{-j\omega t}}{2j}\right]$$

where C is a real constant, ω = angular frequency, t = time and ϕ = phase. Show that

$$y_{ss} = C\sin(\omega t + \phi)$$

24 [*control engineering*] The following transformation is used to derive an equivalent digital filter from an analogue filter:

$$F(s) = \frac{s - 1}{s + 1}$$

where $s = e^{j\omega t}$ and T = sampling period. Show that

$$F(s) = j\tan\left(\frac{\omega T}{2}\right)$$