Chapter 5

SECTION F **Hyperbolic properties**

By the end of this section you will be able to:

- evaluate other hyperbolic functions
- show hyperbolic identities
- understand inverse hyperbolic functions

F1 Other hyperbolic functions

We define hyperbolic functions – cosech, sech and coth – in a similar way to the definitions of trigonometric functions cosec, sec and cot respectively:

5.33 $$\text{cosech}(x) = \frac{1}{\sinh(x)} \quad [\sinh(x) \neq 0]$$

5.34 $$\text{sech}(x) = \frac{1}{\cosh(x)}$$

5.35 $$\coth(x) = \frac{1}{\tanh(x)} = \frac{\cosh(x)}{\sinh(x)} \quad [\sinh(x) \neq 0]$$

Note the similarity with the analogous trigonometric definition:

$$\text{cosec}(x) = \frac{1}{\sin(x)}$$

We use a calculator to evaluate these functions.

Example 23

Determine cosech(0.3), sech(5) and coth(5000).

Solution

By 5.33 we have

$$\text{cosech}(0.3) = \frac{1}{\sinh(0.3)} = [\sinh(0.3)]^{-1}$$

For cosech(0.3), we evaluate $[\sinh(0.3)]^{-1}$ on a calculator. PRESS

(hyp sin 0.3) x^{-1} = which should show 3.283853397.

By using a calculator we have sech(5) = 0.013 and coth (5000) = 1.

F2 Hyperbolic identities

Example 24

Show that

5.36 $\coth^2(x) - 1 = \operatorname{cosech}^2(x)$

Solution

We use the fundamental identity,

5.32 $\cosh^2(x) - \sinh^2(x) = 1$

Dividing both sides of this identity by $\sinh^2(x)$ gives

$$\frac{\cosh^2(x)}{\sinh^2(x)} - \frac{\sinh^2(x)}{\sinh^2(x)} = \frac{1}{\sinh^2(x)}$$

$$\frac{\cosh^2(x)}{\sinh^2(x)} - 1 = \frac{1}{\sinh^2(x)}$$

$$\coth^2(x) - 1 = \operatorname{cosech}^2(x)$$

The last line follows by using $\frac{\cosh(x)}{\sinh(x)} = \coth(x)$ and $\frac{1}{\sinh(x)} = \operatorname{cosech}(x)$.

We can use different variables after the hyperbolic function, it doesn't need to be x. For example $\sinh(A)$.

Note the similarity in the identities of the hyperbolic and trigonometric functions in Table 13.

TABLE 13

Trigonometric	*Hyperbolic*
$\cos^2(A) + \sin^2(A) = 1$	$\cosh^2(A) - \sinh^2(A) = 1$
$\cot^2(A) + 1 = \operatorname{cosec}^2(A)$	$\coth^2(A) - 1 = \operatorname{cosech}^2(A)$
$1 + \tan^2(A) = \sec^2(A)$	$1 - \tanh^2(A) = \operatorname{sech}^2(A)$

There is a technique to move from the trigonometric identity to the analogous hyperbolic identity. We use **Osborne's rule** which says that the trigonometric identity can be replaced by the analogous hyperbolic identity but the sign of any direct (or implied) product of two sinh's must be changed.

For example in trigonometry we have $\cos^2(A) + \sin^2(A) = 1$. Applying Osborne's rule:

$$\cosh^2(A) - \underbrace{\sinh^2(A)}_{\text{direct product of two sinh's}} = 1$$

Remember that $\sinh^2(A) = \sinh(A) \times \sinh(A)$ – so the positive sign (+) in the middle changes to a negative (−) sign.

Similarly in trigonometry: $\cot^2(A) + 1 = \operatorname{cosec}^2(A)$. Using Osborne's rule we have

$$-\coth^2(A) + 1 = -\operatorname{cosech}^2(A) \qquad (*)$$

because $\coth^2(A) = \dfrac{\cosh^2(A)}{\sinh^2(A)}$ and $\operatorname{cosech}^2(A) = \dfrac{1}{\sinh^2(A)}$ – in both cases there is an implied product of two sinh's.

Multiplying both sides of * by −1 gives

$$\coth^2(A) - 1 = \operatorname{cosech}^2(A)$$

This identity is also verified above in **Example 24**.

There are many other hyperbolic identities which can be shown by Osborne's rule. Try verifying some of the following identities:

5.37 $$1 - \tanh^2(A) = \operatorname{sech}^2(A)$$

5.38 $$\cosh(2A) = \cosh^2(A) + \sinh^2(A) = 2\cosh^2(A) - 1 = 1 + 2\sinh^2(A)$$

5.39 $$\sinh(2A) = 2\sinh(A)\cosh(A)$$

5.40 $$\tanh(2A) = \frac{2\tanh(A)}{1 + \tanh^2(A)}$$

5.41 $$\sinh(A \pm B) = \sinh(A)\cosh(B) \pm \cosh(A)\sinh(B)$$

5.42 $$\cosh(A \pm B) = \cosh(A)\cosh(B) \pm \sinh(A)\sinh(B)$$

5.43 $$\tanh(A \pm B) = \frac{\tanh(A) \pm \tanh(B)}{1 \pm \tanh(A)\tanh(B)}$$

5.44 $$\sinh(A) + \sinh(B) = 2\sinh\left(\frac{A + B}{2}\right)\cosh\left(\frac{A - B}{2}\right)$$

5.45 $$\sinh(A) - \sinh(B) = 2\cosh\left(\frac{A + B}{2}\right)\sinh\left(\frac{A - B}{2}\right)$$

5.46 $$\cosh(A) + \cosh(B) = 2\cosh\left(\frac{A + B}{2}\right)\cosh\left(\frac{A - B}{2}\right)$$

5.47 $$\cosh(A) - \cosh(B) = 2\sinh\left(\frac{A + B}{2}\right)\sinh\left(\frac{A - B}{2}\right)$$

For example, to show 5.41 :

$$\sinh(A + B) = \sinh(A)\cosh(B) + \cosh(A)\sinh(B)$$

Notice that there is **no** direct or implied product of two sinh's, thus the hyperbolic identity is the same as the trigonometric identity:

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

By Osborne's rule:

$$\sinh(A + B) = \sinh(A)\cosh(B) + \cosh(A)\sinh(B)$$

F3 Inverse hyperbolic functions

The inverse hyperbolic functions of $\sinh(x)$, $\cosh(x)$ and $\tanh(x)$ are denoted by $\sinh^{-1}(x)$, $\cosh^{-1}(x)$ and $\tanh^{-1}(x)$ respectively.

These functions are sometimes designated by arsinh, arcosh and artanh.

? **What does $\sinh^{-1}$ represent?**

If $\sinh(y) = x$ then

$$y = \sinh^{-1}(x)$$

(The following are correct to 3 d.p.) For example, $\sinh(2.1) = 4.022$ therefore

$$\sinh^{-1}(4.022) = 2.1$$

? **What is $\sinh^{-1}(3.627)$, given that $\sinh(2) = 3.627$?**

$$\sinh^{-1}(3.627) = 2$$

Similarly if $\cosh(y) = x$ then

$$y = \cosh^{-1}(x) \qquad (x \geq 1)$$

The domain of inverse cosh function is $x \geq 1$.

? **What is $\cosh^{-1}(1)$ equal to, given that $\cosh(0) = 1$?**

$$\cosh^{-1}(1) = 0$$

From $\tanh(y) = x$ it follows that

$$y = \tanh^{-1}(x) \qquad (-1 < x < 1)$$

The domain of the inverse tanh lies between -1 and $+1$, that is $-1 < x < 1$.

To evaluate these inverse functions we can use a calculator.

Example 25

Determine, correct to three d.p., $\sinh^{-1}(3)$, $\sinh^{-1}(-3)$, $\cosh^{-1}(3)$, $\tanh^{-1}(0)$, $\tanh^{-1}(0.25)$ and $\tanh^{-1}(1)$.

Solution

Using a calculator to evaluate $\sinh^{-1}(3)$, PRESS **hyp** **SHIFT** **sin** **3** **=** which should show 1.818446459.

So $\sinh^{-1}(3) = 1.818$. Similarly we have:

? $\sinh^{-1}(-3) = -1.818$, $\cosh^{-1}(3) = 1.763$, $\tanh^{-1}(0) = 0$, $\tanh^{-1}(0.25) = 0.255$ and for $\tanh^{-1}(1)$, the calculator shows an error. **Why?**

The function $\tanh^{-1}(x)$ is only valid for x between -1 and $+1$ and is not a real number for $x \geq 1$ or $x \leq -1$. (See Fig. 14**c** below.)

You can plot the inverse hyperbolic functions on a graphical calculator or a computer algebra system (Fig. 14).

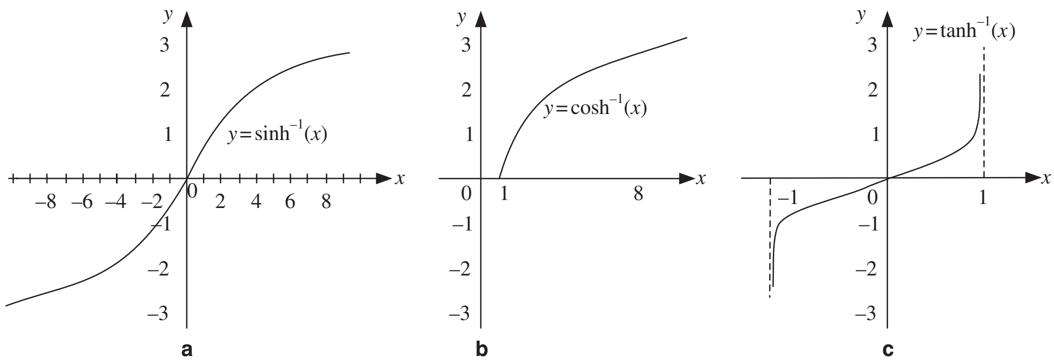

Fig. 14

? **Do you notice why we cannot evaluate $\tanh^{-1}(1)$?**

There is a vertical asymptote at $x = 1$. Similarly we cannot evaluate $\tanh^{-1}(-1)$.

As can be seen by the graph of Fig. 14**b**, the inverse cosh function, $\cosh^{-1}$, is only valid for x greater than or equal to 1. If we try to evaluate $\cosh^{-1}(x)$ for x values less than 1, the calculator shows an error.

Example 26 *mechanics*

The length, s, of a cable can be found from

$$x = \frac{T}{w}\sinh^{-1}\left(\frac{sw}{T}\right) \qquad (*)$$

where T is tension, w is load per unit length and x is horizontal distance. Show that

$$s = \frac{T}{w}\sinh\left(\frac{wx}{T}\right)$$

Solution

Multiplying both sides of the given equation, (*), by w gives

$$wx = T\sinh^{-1}\left(\frac{sw}{T}\right)$$

We need to obtain s from the Right-Hand Side. Divide both sides by T:

$$\frac{wx}{T} = \sinh^{-1}\left(\frac{sw}{T}\right)$$

How do we remove $\sinh^{-1}$?

Take sinh of both sides:

$$\sinh\left(\frac{wx}{T}\right) = \sinh\left[\sinh^{-1}\left(\frac{sw}{T}\right)\right] = \frac{sw}{T}$$

(because $\sinh^{-1}$ is the inverse function of sinh).

Transposing to make s the subject gives $s = \dfrac{T}{w}\sinh\left(\dfrac{wx}{T}\right)$.

SUMMARY

The hyperbolic identities can be established from the analogous trigonometric identities by using Osborne's rule which says that the sign of the product of two sinh's must be changed.

Inverse hyperbolic functions are denoted by $\sinh^{-1}$, $\cosh^{-1}$ and $\tanh^{-1}$. We can evaluate these functions on a calculator.

Exercise 5(f)

Solutions are given at the end of this additional material. Complete solutions are in this website.

1 Evaluate sech(2), cosech(2) and coth(10).

2 Find $\sinh^{-1}(\pi)$, $\sinh^{-1}(-\pi)$, $\tanh^{-1}(0)$, $\tanh^{-1}(0.5)$, $\cosh^{-1}(\pi)$, $\cosh^{-1}(1000)$ and $\cosh^{-1}(0)$.

3 Without using a calculator, determine

$\sinh[\sinh^{-1}(\pi)]$, $\sinh[\sinh^{-1}(5)]$,

$\cosh[\cosh^{-1}(\pi)]$ and $\tanh[\tanh^{-1}(0.236)]$

4 Find x which satisfies

a $\cosh(x) = 1.7$

b $\sinh(x) = \pi$

c $\tanh(x) = 0.5$

5 Without using Osborne's rule, show that

a $1 - \tanh^2(x) = \text{sech}^2(x)$

b $2\sinh(x)\cosh(x) = \sinh(2x)$

Use a computer algebra system or a graphical calculator for question 6.

6 Plot on different axes the following graphs for x between -10 and 10:

a $y = \text{sech}(x)$

b $y = \coth(x)$

c $y = \text{cosech}(x)$

7 Show that

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x)$$

8 Show that

$$\cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 1 + 2\sinh^2(x)$$

9 Without using Osborne's rule, show that

$$2\sinh\left(\frac{A+B}{2}\right)\cosh\left(\frac{A-B}{2}\right) = \sinh(A) + \sinh(B)$$

10 *[mechanics]* The length, s, of a cable with span L and sag h can be determined by

$$s = \frac{L}{2}\left\{\left[1 + \left(\frac{4h}{L}\right)^2\right]^{1/2} + \left(\frac{L}{4h}\right)\sinh^{-1}\left(\frac{4h}{L}\right)\right\}$$

Find the length of the cable which has a span of 200 m and a sag of 60 m.

11 *[mechanics]* The length, s, of a cable can be evaluated from the equation

$$x = \frac{T}{w}\sinh^{-1}\left[\frac{ws}{T} + \tan^{-1}(\theta)\right]$$

where T represents horizontal tension, w is load per unit length, $\tan^{-1}(\theta)$ is an angle and x is horizontal distance. Make s the subject of the equation.

12 *[electrical principles]* A transmission line of length L has an impedance Z given by

$$Z = \frac{2Z_0 e^{-\gamma L}}{(1 + e^{-\gamma L})(1 - e^{-\gamma L})}$$

where Z_0 is the characteristic impedance and γ is the propagation coefficient. Show that

$$Z = Z_0\,\text{cosech}(\gamma L)$$

Miscellaneous exercise **5 (extra)**

Solutions are given at the end of this additional material. Complete solutions are in this website.

16 Without using Osborne's rule, show that

$$\tanh(x + y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)}$$

17 [*electrical principles*] In a symmetrical network we have the following equations:

$$Z_1 + Z_2 = 2Z_0\coth(\gamma L) \quad (*)$$

$$\frac{2Z_1Z_2}{Z_1 + Z_2} = Z_0\tanh(\gamma L) \quad (**)$$

Show that

$$Z_2 = Z_0[\coth(\gamma L) \pm \operatorname{cosec}(\gamma L)]$$

(Z_0, Z_1 and Z_2 are impedences, γ is the propagation coefficient and L is the length.)

For question 18 use a computer algebra system (or a graphical calculator).

18 [*mechanics*] The length, s, of a cable with span L and sag h is given by

$$s = \frac{L}{2}\left\{\left[1 + \left(\frac{4h}{L}\right)^2\right]^{1/2} + \left(\frac{L}{4h}\right)\sinh^{-1}\left(\frac{4h}{L}\right)\right\}$$

a Plot the graph of s for $-60 \le h \le 0$ with $L = 200$ m.

b Determine h, if $L = 200$ m and $s = 240.87$ m.

19 [*electrical principles*] A transmission line of length L has a sending end voltage V_s and sending end current I_s given by

$$V_s = V\cosh(\gamma L) + IZ\sinh(\gamma L) \quad (\dagger)$$

$$I_s = I\cosh(\gamma L) + \frac{V}{Z}\sinh(\gamma L) \quad (\dagger\dagger)$$

where V is receiving end voltage, I is receiving end current, Z is characteristic impedance and γ is propagation coefficient. Show that

$$I = I_s\cosh(\gamma L) - \frac{V_s}{Z}\sinh(\gamma L)$$

$$V = V_s\cosh(\gamma L) - ZI_s\sinh(\gamma L)$$