9. First we obtain $\mathbf{AB}=\begin{pmatrix}-5 & 6\\ 2 & 3\end{pmatrix}\begin{pmatrix}1\\ -4\end{pmatrix}=\begin{pmatrix}-29\\ -10\end{pmatrix}$

Using $\mathbf{C}_T=(\mathbf{B}\quad \mathbf{AB})$ gives $\mathbf{C}_T=\begin{pmatrix}1 & -29\\ -4 & -10\end{pmatrix}$

To find the observability matrix we need to determine $\mathbf{CA}$

$$\mathbf{CA}=(2\quad 9)\begin{pmatrix}-5 & 6\\ 2 & 3\end{pmatrix}=\left(\left(2\times(-5)\right)+(9\times 2)\quad (2\times 6)+(9\times 3)\right)=(8\quad 39)$$

Substituting $\mathbf{CA}=(8\quad 39)$ and $\mathbf{C}=(2\quad 9)$ into $\mathbf{O}_T=\begin{pmatrix}\mathbf{C}\\ \mathbf{CA}\end{pmatrix}$ gives $\mathbf{O}_T=\begin{pmatrix}2 & 9\\ 8 & 39\end{pmatrix}$.

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10. We first find **AB**

$$\mathbf{AB}=\begin{pmatrix}2 & 5\\ -1 & 3\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}=\begin{pmatrix}7\\ 2\end{pmatrix}$$

Applying (11.6)

$$\mathbf{C}_T=\begin{pmatrix}1 & 7\\ 1 & 2\end{pmatrix}$$

To determine the observability matrix we have to find **CA**

$$\mathbf{CA}=(-1\quad 4)\begin{pmatrix}2 & 5\\ -1 & 3\end{pmatrix}=(-6\quad 7)$$

By (11.7)

$$\mathbf{O}_T=\begin{pmatrix}-1 & 4\\ -6 & 7\end{pmatrix}$$

(i) By (11.1)

$$\det\begin{pmatrix}1 & 7\\ 1 & 2\end{pmatrix}=(1\times 2)-(1\times 7)=-5\neq 0$$

Thus the system is controllable.

(ii) Similarly

$$\det\begin{pmatrix}-1 & 4\\ -6 & 7\end{pmatrix}=(-1\times 7)-(-6\times 4)=17\neq 0$$

The system is observable.

11.(a) Similar to question 2.

$$\mathbf{C}_T=\begin{pmatrix}-1 & 11\\ 5 & 6\end{pmatrix}\quad\text{and}\quad \det\mathbf{C}_T=-6-(5\times 11)\neq 0$$

Hence the system is controllable. Also the system is observable

$$\mathbf{O}_T=\begin{pmatrix}3 & 0\\ -3 & 6\end{pmatrix}\quad\text{and}\quad \det\mathbf{O}_T=(6\times 3)-0\neq 0$$

(11.1) $\det\begin{pmatrix}a & b\\ c & d\end{pmatrix}=ad-cb$

(11.6) $\mathbf{C}_T=(\mathbf{B}\quad \mathbf{AB})$

(11.7) $\mathbf{O}_T=\begin{pmatrix}\mathbf{C}\\ \mathbf{CA}\end{pmatrix}$

(b) We have

$$\mathbf{C}_T = \begin{pmatrix} 1 & 4 \\ -1 & -4 \end{pmatrix} \text{ and } \det \mathbf{C}_T = -4 + 4 = 0$$

The system is not controllable. Similarly we find the system is observable because

$$\mathbf{O}_T = \begin{pmatrix} 3 & 2 \\ 20 & 16 \end{pmatrix} \text{ and } \det \mathbf{O}_T = (3\times 16) - (20\times 2) \neq 0$$

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12. (a) $s\mathbf{I} - \mathbf{A} = s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} s+1 & -3 \\ -2 & s-1 \end{pmatrix}$

By (11.1)

$$\det(s\mathbf{I} - \mathbf{A}) = (s+1)(s-1) - 6 = s^2 - 1 - 6$$

Hence $s^2 - 7 = 0$ which gives $s = \pm\sqrt{7}$. System poles are $s_1 = \sqrt{7}$, $s_2 = -\sqrt{7}$

(b) $s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 2 & 10 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} s-2 & -10 \\ -5 & s+3 \end{pmatrix}$

By (11.1)

$$\det\begin{pmatrix} s-2 & -10 \\ -5 & s+3 \end{pmatrix} = (s-2)(s+3) - 50 = s^2 + s - 6 - 50 = s^2 + s - 56$$

Need to solve

$$s^2 + s - 56 = 0$$
$$(s-7)(s+8) = 0$$

System poles are $s_1 = 7$, $s_2 = -8$

(c) We have $s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s+5.1 & -2.2 \\ -3.7 & s-6.1 \end{pmatrix}$

Using (11.1)

$$\det\begin{pmatrix} s+5.1 & -2.2 \\ -3.7 & s-6.1 \end{pmatrix} = (s+5.1)(s-6.1) - (3.7\times 2.2) = s^2 - s - 31.11 - 8.14$$

We have the equation

$$s^2 - s - 39.25 = 0$$

Using quadratic formula (1.16) with $a = 1$, $b = -1$ and $c = -39.25$ gives

$$s = \frac{1 \pm \sqrt{1 - (4\times 1\times(-39.25))}}{2} = \frac{1 \pm \sqrt{158}}{2} = 6.78,\ -5.78$$

System poles are $s_1 = 6.78$, $s_2 = -5.78$

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13. We have

$$\det(s\mathbf{I} - \mathbf{A}) = \det\begin{pmatrix} s + \frac{R}{L} & \frac{1}{L} \\ -\frac{1}{C} & s \end{pmatrix} \underbrace{=}_{\text{by (11.1)}} s\left(s + \frac{R}{L}\right) + \frac{1}{LC} = s^2 + \frac{R}{L}s + \frac{1}{LC}$$

(1.16) $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (11.1) $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$

For poles

$$s^2+\frac{R}{L}s+\frac{1}{LC}=0$$

Multiplying by LC gives

$$LCs^2+RCs+1=0$$

To find s we use (1.16) with $a=LC,\ b=RC$ and $c=1$

$$s=\frac{-RC\pm\sqrt{R^2C^2-4LC}}{2LC}$$

The system poles are

$$s_1=\frac{-RC+\sqrt{R^2C^2-4LC}}{2LC},\ s_2=\frac{-RC-\sqrt{R^2C^2-4LC}}{2LC}$$

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14. We have

$$(s\mathbf{I}-\mathbf{A})=s\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}0&1\\-k/m&-c/m\end{pmatrix}=\begin{pmatrix}s&0\\0&s\end{pmatrix}-\begin{pmatrix}0&-1\\-k/m&-c/m\end{pmatrix}=\begin{pmatrix}s&1\\k/m&s+c/m\end{pmatrix}$$

$$\det(s\mathbf{I}-\mathbf{A})=\det\begin{pmatrix}s&-1\\k/m&s+c/m\end{pmatrix}\underbrace{=}_{\text{by (11.1)}}s\left(s+\frac{c}{m}\right)+\frac{k}{m}$$

For poles this determinant is equal to zero

$$s\left(s+\frac{c}{m}\right)+\frac{k}{m}=0$$

$$s^2+\frac{c}{m}s+\frac{k}{m}=0$$

Multiplying by m

$$ms^2+cs+k=0$$

Using the quadratic formula gives

$$s=\frac{-c\pm\sqrt{c^2-4mk}}{2m}$$

The system poles are

$$s_1=\frac{-c+\sqrt{c^2-4mk}}{2m},\quad s_2=\frac{-c-\sqrt{c^2-4mk}}{2m}$$

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15. (a) We first find the system poles:

$$s\mathbf{I}-\mathbf{A}=\begin{pmatrix}s&0\\0&s\end{pmatrix}-\begin{pmatrix}-1&-3\\5&2\end{pmatrix}=\begin{pmatrix}s+1&3\\-5&s-2\end{pmatrix}$$

By (11.1)

$$\det\begin{pmatrix}s+1&3\\-5&s-2\end{pmatrix}=(s+1)(s-2)-(-15)=s^2-s-2+15=s^2-s+13$$

We need to solve

$$s^2-s+13=0$$

(1.16) $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

(11.1) $$\det\begin{pmatrix}a&b\\c&d\end{pmatrix}=ad-cb$$

Using the quadratic formula with $a=1,\ b=-1$ and $c=13$

$$s=\frac{1\pm\sqrt{1-(4\times1\times13)}}{2}$$
$$=\frac{1}{2}\pm\frac{\sqrt{-51}}{2}=\frac{1}{2}\pm j\frac{\sqrt{51}}{2}$$

The real part of s, $\mathrm{Re}(s)=\frac{1}{2}>0$. Thus the system is not stable.

(b)

$$s\mathbf{I}-\mathbf{A}=\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}-\begin{pmatrix} -6 & -1 \\ 9 & -3 \end{pmatrix}=\begin{pmatrix} s+6 & 1 \\ -9 & s+3 \end{pmatrix}$$

By (11.1)

$$\det\begin{pmatrix} s+6 & 1 \\ -9 & s+3 \end{pmatrix}=(s+6)(s+3)-(-9\times1)=s^2+9s+18+9=s^2+9s+27$$

We need to solve $s^2+9s+27=0$

$$s=\frac{-9\pm\sqrt{81-(4\times1\times27)}}{2}=-\frac{9}{2}\pm\frac{\sqrt{-27}}{2}=-\frac{9}{2}\pm\frac{j\sqrt{27}}{2}$$

$\mathrm{Re}(s)=-\frac{9}{2}<0$. The system is stable.

(c) $s\mathbf{I}-\mathbf{A}=\begin{pmatrix} s-3.1 & -6.5 \\ -1.7 & s-4.8 \end{pmatrix}$

Using (11.1)

$$\det\begin{pmatrix} s-3.1 & -6.5 \\ -1.7 & s-4.8 \end{pmatrix}=(s-3.1)(s-4.8)-(1.7\times6.5)$$
$$=s^2-7.9s+14.88-11.05$$
$$=s^2-7.9s+3.83$$

We need to solve $s^2-7.9s+3.83=0$. Applying the quadratic formula

$$s=\frac{7.9\pm\sqrt{7.9^2-(4\times1\times3.83)}}{2}$$
$$=\frac{7.9\pm\sqrt{47.09}}{2}$$
$$s_1=7.381,\qquad s_2=0.510$$

Since $\mathrm{Re}(s)>0$, the system is unstable.

(11.1) $$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}=ad-cb$$