Complete solutions to Exercise 11(h)

1. Using (11.21) on each of the nodes:

Node 1; $300 + 600 + T_2 + T_3 = 4T_1$

Node 2; $T_1 + 600 + 500 + T_4 = 4T_2$

Node 3; $300 + T_1 + T_4 + 350 = 4T_3$

Node 4; $T_3 + T_2 + 500 + 350 = 4T_4$

Simplifying the above

$$\begin{aligned} 4T_1 - T_2 - T_3 \qquad &= 900 \\ -T_1 + 4T_2 \qquad - T_4 &= 1100 \\ -T_1 \qquad + 4T_3 - T_4 &= 650 \\ - T_2 - T_3 + 4T_4 &= 850 \end{aligned}$$

In matrix form we have

$$\begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{pmatrix} = \begin{pmatrix} 900 \\ 1100 \\ 650 \\ 850 \end{pmatrix}$$

Using a graphical calculator should indicate the following results:

$$T_1 = 443.75K,\ T_2 = 493.75K,\ T_3 = 381.25 \text{ and } T_4 = 431.25K$$

Yes, because T_2 is closest to the hottest surfaces, $600K$ and $500K$. Thus T_2 is the highest temperature. Similarly T_3 is closest to the coldest surfaces $300K$ and $350K$, hence T_3 is the lowest temperature.

---

2. (i) All four points 1, 2, 3 and 4 are interior nodes, so we use (11.21):

Node 1; $400 + 500 + T_2 + T_3 = 4T_1$

Node 2; $T_1 + 500 + 400 + T_4 = 4T_2$

Node 3; $400 + T_1 + T_4 + 400 = 4T_3$

Node 4; $T_3 + T_2 + 400 + 400 = 4T_4$

Rearranging and simplifying these equations we obtain

$$\begin{aligned} 4T_1 - T_2 - T_3 \qquad &= 900 \\ -T_1 + 4T_2 \qquad - T_4 &= 900 \\ -T_1 \qquad + 4T_3 - T_4 &= 800 \\ - T_2 - T_3 + 4T_4 &= 800 \end{aligned} \qquad (*)$$

In matrix form we have

$$\begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{pmatrix} = \begin{pmatrix} 900 \\ 900 \\ 800 \\ 800 \end{pmatrix}$$

Using a graphical calculator gives

$$T_1 = 437.5K,\ T_2 = 437.5K,\ T_3 = 412.5K \text{ and } T_4 = 412.5K$$

(11.21) $$T_a + T_b + T_d + T_e = 4T_c$$

(ii) $T_1 = T_2$ and $T_3 = T_4$. This is because we have symmetry - the temperature of the left nodal points is equal to the right nodal points.

(iii) Yes, by substituting $T_1 = T_2$ and $T_3 = T_4$ into (*).

3. Since all are interior nodes we use (11.21):

Node 1; $800 + 650 + T_2 + T_4 = 4T_1$

Node 2; $T_1 + 650 + T_3 + T_5 = 4T_2$

Node 3; $T_2 + 650 + 600 + T_6 = 4T_3$

Node 4; $800 + T_1 + T_5 + T_7 = 4T_4$

Node 5; $T_4 + T_2 + T_6 + T_8 = 4T_5$

Node 6; $T_5 + T_3 + 600 + T_9 = 4T_6$

Node 7; $800 + T_4 + T_8 + 950 = 4T_7$

Node 8; $T_7 + T_5 + T_9 + 950 = 4T_8$

Node 9; $T_8 + T_6 + 600 + 950 = 4T_9$

Rearranging and simplifying gives

$$\begin{array}{cccccccccccc}
4T_1 & -T_2 & & -T_4 & & & & & & = & 1450 \\
-T_1 & +4T_2 & -T_3 & & -T_5 & & & & & = & 650 \\
 & -T_2 & +4T_3 & & & -T_6 & & & & = & 1250 \\
-T_1 & & & +4T_4 & -T_5 & & -T_7 & & & = & 800 \\
 & -T_2 & & -T_4 & +4T_5 & -T_6 & & -T_8 & & = & 0 \\
 & & -T_3 & & -T_5 & +4T_6 & & & -T_9 & = & 600 \\
 & & & -T_4 & & & +4T_7 & -T_8 & & = & 1750 \\
 & & & & -T_5 & & -T_7 & +4T_8 & -T_9 & = & 950 \\
 & & & & & -T_6 & & -T_8 & +4T_9 & = & 1550
\end{array}$$

In matrix form we have

$$\begin{pmatrix}
4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4
\end{pmatrix}
\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{pmatrix}
=
\begin{pmatrix} 1450 \\ 650 \\ 1250 \\ 800 \\ 0 \\ 600 \\ 1750 \\ 950 \\ 1550 \end{pmatrix}$$

By using MAPLE we obtain the following temperatures (2 d.p.) in K:
$T_1 = 732.14$, $T_2 = 698.21$, $T_3 = 660.71$, $T_4 = 780.36$, $T_5 = 750$, $T_6 = 694.64$, $T_7 = 839.29$, $T_8 = 826.79$ and $T_9 = 767.86$.

For Maple commands see next page.

(11.21) $T_a + T_b + T_d + T_e = 4T_c$

```
> A:=matrix([[4,-1,0,-1,0,0,0,0,0],[-1,4,-1,0,-1,0,0,0,0],[0,-
1,4,0,0,-1,0,0,0],[-1,0,0,4,-1,0,-1,0,0],[0,-1,0,-1,4,-1,0,-
1,0],[0,0,-1,0,-1,4,0,0,-1],[0,0,0,-1,0,0,4,-1,0],[0,0,0,0,-1,0,-
1,4,-1],[0,0,0,0,0,-1,0,-1,4]]);
```

$$A := \begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix}$$

```
> b:=vector([1450,650,1250,800,0,600,1750,950,1550]);
```

$$b := [1450, 650, 1250, 800, 0, 600, 1750, 950, 1550]$$

```
> with(linalg):
> linsolve(A,b);
```

$$\left[\frac{5125}{7}, \frac{9775}{14}, \frac{4625}{7}, \frac{10925}{14}, 750, \frac{9725}{14}, \frac{5875}{7}, \frac{11575}{14}, \frac{5375}{7}\right]$$

```
> evalf(%,5);
```

$$[732.14, 698.21, 660.71, 780.36, 750., 694.64, 839.29, 826.79, 767.86]$$

---

4. Let $T_1, T_2, \ldots, T_9$ and T_{10} represent the temperature at nodes 1, 2, 3,..., 9 and 10 respectively. Note that points 9 and 10 are exterior points.
For interior nodes 1, 2, 3,..., 8 we use (11.21):
Node 1; $600 + 600 + T_3 + T_2 = 4T_1$
Node 2; $600 + T_1 + T_4 + T_1 = 4T_2$
Node 3; $T_1 + 600 + T_5 + T_4 = 4T_3$
Node 4; $T_2 + T_3 + T_6 + T_3 = 4T_4$
Node 5; $T_3 + 600 + T_7 + T_6 = 4T_5$
Node 6; $T_4 + T_5 + T_8 + T_5 = 4T_6$
Node 7; $T_5 + 600 + T_9 + T_8 = 4T_7$
Node 8; $T_6 + T_7 + T_{10} + T_7 = 4T_8$
For the exterior nodes we use (11.20):

Node 9; $600 + 2T_7 + T_{10} + \left(\frac{2\times 10\times 0.1}{1}\right)400 = 2\left(\frac{10\times 0.1}{1} + 2\right)T_9$

Node 10; $T_9 + 2T_8 + T_9 + \left(\frac{2\times 10\times 0.1}{1}\right)400 = 2\left(\frac{10\times 0.1}{1} + 2\right)T_{10}$

Simplifying and rearranging gives

(11.20) $$T_a + 2T_c + T_b + \frac{2h\Delta x}{k}(T_\infty) = 2\left(\frac{h\Delta x}{k} + 2\right)T_{ex}$$

$$
\begin{array}{cccccccccccc}
4T_1 & -T_2 & -T_3 & & & & & & & & = & 1200 \\
-2T_1 & +4T_2 & & -T_4 & & & & & & & = & 600 \\
-T_1 & & +4T_3 & -T_4 & -T_5 & & & & & & = & 600 \\
 & -T_2 & -2T_3 & +4T_4 & & -T_6 & & & & & = & 0 \\
 & & -T_3 & & +4T_5 & -T_6 & -T_7 & & & & = & 600 \\
 & & & -T_4 & -2T_5 & +4T_6 & & -T_8 & & & = & 0 \\
 & & & & -T_5 & & +4T_7 & -T_8 & -T_9 & & = & 600 \\
 & & & & & -T_6 & -2T_7 & +4T_8 & & -T_{10} & = & 0 \\
 & & & & & & -2T_7 & & +6T_9 & -T_{10} & = & 1400 \\
 & & & & & & & -2T_8 & -2T_9 & +6T_{10} & = & 800
\end{array}
$$

Writing these equations in matrix form we obtain

$$
\begin{pmatrix}
4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -2 & 4 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 4 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -2 & 4 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 4 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -2 & 4 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 6 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 6
\end{pmatrix}
\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \\ T_{10} \end{pmatrix}
=
\begin{pmatrix} 1200 \\ 600 \\ 600 \\ 0 \\ 600 \\ 0 \\ 600 \\ 0 \\ 1400 \\ 800 \end{pmatrix}
$$

By using MAPLE and rounding off to 2 d.p. we have (in K)

$T_1 = 596.33,\ T_2 = 594.84,\ T_3 = 590.49,\ T_4 = 586.71,\ T_5 = 578.91,\ T_6 = 571.03$,

$T_7 = 554.10,\ T_8 = 539.60,\ T_9 = 497.89$ and $T_{10} = 479.16$.

The MAPLE commands are similar to solution 3.

There are no other solutions because

$$\det \mathbf{A} = 507106 \neq 0$$

So by (11.13) the above solution is unique.

---

5. The first 10 nodes are interior nodes so we use (11.21):

Node 1; $350 + 350 + T_2 + T_3 = 4T_1$

Node 2; $T_1 + 350 + T_1 + T_4 = 4T_2$

Node 3; $350 + T_1 + T_4 + T_5 = 4T_3$

Node 4; $T_3 + T_2 + T_3 + T_6 = 4T_4$

Node 5; $350 + T_3 + T_6 + T_7 = 4T_5$

Node 6; $T_5 + T_4 + T_5 + T_8 = 4T_6$

Node 7; $350 + T_5 + T_8 + T_9 = 4T_7$

Node 8; $T_7 + T_6 + T_7 + T_{10} = 4T_8$

Node 9; $350 + T_7 + T_{10} + T_{11} = 4T_9$

Node 10; $T_9 + T_8 + T_9 + T_{12} = 4T_{10}$

(11.21) $$T_a + T_b + T_d + T_e = 4T_c$$

11 and 12 are exterior nodes so we apply (11.20).

Node 11; $350 + 2T_9 + T_{12} + \underbrace{(2\times 50\times 400\times 0.025)}_{=1000} = 2\underbrace{((50\times 0.025)+2)}_{=6.5}T_{11}$

Node 12; $T_{11} + 2T_{10} + T_{11} + \underbrace{(2\times 50\times 400\times 0.025)}_{=1000} = 2\underbrace{((50\times 0.025)+2)}_{=6.5}T_{12}$

Rearranging and simplifying

$$
\begin{aligned}
4T_1 - T_2 - T_3 &= 700\\
-2T_1 + 4T_2 - T_4 &= 350\\
-T_1 + 4T_3 - T_4 - T_5 &= 350\\
-T_2 - 2T_3 + 4T_4 - T_6 &= 0\\
-T_3 + 4T_5 - T_6 - T_7 &= 350\\
-T_4 - 2T_5 + 4T_6 - T_8 &= 0\\
-T_5 + 4T_7 - T_8 - T_9 &= 350\\
-T_6 - 2T_7 + 4T_8 - T_{10} &= 0\\
-T_7 + 4T_9 - T_{10} - T_{11} &= 350\\
-T_8 - 2T_9 + 4T_{10} - T_{12} &= 0\\
-2T_9 + 6.5T_{11} - T_{12} &= 1350\\
-2T_{10} - 2T_{11} + 6.5T_{12} &= 1000
\end{aligned}
$$

Writing this in matrix form:

$$
\begin{pmatrix}
4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-2 & 4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & -2 & 4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 0 & 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & -2 & 4 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 0 & 4 & -1 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & -2 & 4 & 0 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 4 & -1 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 4 & 0 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 6.5 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 6.5
\end{pmatrix}
\begin{pmatrix} T_1\\ T_2\\ T_3\\ T_4\\ T_5\\ T_6\\ T_7\\ T_8\\ T_9\\ T_{10}\\ T_{11}\\ T_{12} \end{pmatrix}
=
\begin{pmatrix} 700\\ 350\\ 350\\ 0\\ 350\\ 0\\ 350\\ 0\\ 350\\ 0\\ 1350\\ 1000 \end{pmatrix}
$$

By using MAPLE we obtain the following temperatures (in K):

$T_1 = 350.48,\ T_2 = 350.67,\ T_3 = 351.23,\ T_4 = 351.73,\ T_5 = 352.72,\ T_6 = 353.80$

$T_7 = 355.85,\ T_8 = 358.02,\ T_9 = 362.66,\ T_{10} = 366.59,\ T_{11} = 378.21$ and $T_{12} = 383.01$

(11.20) $$T_a + 2T_c + T_b + \frac{2h\Delta x}{k}(T_\infty) = 2\left(\frac{h\Delta x}{k} + 2\right)T_{ex}$$

The MAPLE commands are:

```
> A:=matrix([[4, -1, -1, 0,0,0,0,0,0,0,0,0], [-2, 4, 0, -1,0,0,0,0,0,0,0,0], [-1, 0, 4, -1,-1,0,0,0,0,0,0,0], [0, -1, -2, 4,0,-1,0,0,0,0,0,0],[0, 0, -1, 0,4,-1,-1,0,0,0,0,0],[0,0,0,-1,-2,4,0,-1,0,0,0,0],[0,0,0,0,-1,0,4,-1,-1,0,0,0],[0,0,0,0,0,-1,-2,4,0,-1,0,0],[0,0,0,0,0,0,-1,0,4,-1,-1,0],[0,0,0,0,0,0,0,-1,-2,4,0,-1],[0,0,0,0,0,0,0,0,-2,0,6.5,-1],[0,0,0,0,0,0,0,0,0,-2,-2,6.5]]);
```

$$A := \begin{bmatrix} 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -2 & 4 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 4 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -2 & 4 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 4 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 6.5 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 6.5 \end{bmatrix}$$

```
> b:=vector([700,350,350,0,350,0,350,0,350,0,1350,1000]);
```

$$b := [700, 350, 350, 0, 350, 0, 350, 0, 350, 0, 1350, 1000]$$

```
> with(linalg):
> linsolve(A,b);
```

[350.4761472, 350.6716694, 351.2329190, 351.7343832, 352.7211457, 353.8000256, 355.8516385, 358.0234275, 362.6619808, 366.5904077, 378.2058773, 383.0142414]

```
> evalf(%,5);
```

[350.48, 350.67, 351.23, 351.73, 352.72, 353.80, 355.85, 358.02, 362.66, 366.59, 378.21, 383.01]