Complete Solutions to Exercise 16(g)

1. (i) The mean \( \mu = E(T) \) and the formula is given by

\[
\mu = E(T) = \int_0^1 t \left( \frac{t}{50} \right) dt = \frac{1}{50} \int_0^1 (t^2) dt \quad \text{[Taking out } \frac{1}{50} \text{]}
\]

\[
= \frac{1}{50} \left[ \frac{t^3}{3} \right]_0^1 \quad \text{[Integrating]}
\]

\[
= \frac{1}{150} [10^3 - 0^3] = 6.67
\]

(ii) \( E(T^2) \) can be evaluated by using the formula \( E(T^2) = \int_0^1 t^2 f(t) dt \). We are given that

\[
f(t) = \frac{t}{50} \quad \text{for } t \text{ between } 0 \text{ and } 10 \therefore
\]

\[
E(T^2) = \int_0^1 t^2 \left( \frac{t}{50} \right) dt = \frac{1}{50} \int_0^1 (t^3) dt
\]

\[
= \frac{1}{50} \left[ \frac{t^4}{4} \right]_0^1 = \frac{1}{200} [10^4 - 0^4] = 50
\]

(iii) The variance \( \sigma^2 \) is given by the formula \( \sigma^2 = E(T^2) - \mu^2 \). Substituting the results of parts (i) and (ii), \( \mu = 6.67 \) and \( E(T^2) = 50 \), into this formula:

\[
\sigma^2 = E(T^2) - \mu^2 = 50 - 6.67^2 = 5.5111
\]

(iv) The probability that the waiting time is at most 5 minutes is given by

\[
P(0 \leq T \leq 5) = \int_0^5 t \left( \frac{t}{50} \right) dt = \frac{1}{50} \int_0^5 (t^2) dt
\]

\[
= \frac{1}{50} \left[ \frac{t^3}{3} \right]_0^5 = \frac{1}{100} [5^3 - 0^3] = \frac{25}{100} = \frac{1}{4}
\]

(v) Using the result of part (iv) we have that the probability of the waiting time is more than 5 minutes is \( 1 - \frac{1}{4} = \frac{3}{4} \).

2. (a) The probability that the television breaks down within a year is given by

\[
P(0 \leq T \leq 1) = \int_0^1 \left( \frac{t^3}{2} \right) dt
\]

\[
= \frac{1}{2} \int_0^1 t^3 dt = \frac{1}{2} \left[ \frac{t^4}{4} \right]_0^1 = \frac{1}{4} \times \frac{1}{2} [1^4 - 0^4] = \frac{1}{4096}
\]

(b) Similarly the probability between the 1\textsuperscript{st} and 2\textsuperscript{nd} year is

\[
P(1 \leq T \leq 2) = \int_1^2 \left( \frac{t^3}{2} \right) dt
\]

\[
= \frac{1}{2} \int_1^2 t^3 dt = \frac{1}{2} \left[ \frac{t^4}{4} \right]_1^2 = \frac{1}{4} \times \frac{1}{2} [2^4 - 1^4] = \frac{15}{4096}
\]
(c) The probability of the television breaking down after 6 years is \( P(6 \leq T \leq 8) \) because the television does not last more than 8 years. Thus we have

\[
P(6 \leq T \leq 8) = \int_{6}^{8} \frac{t^3}{2400} \, dt = \frac{1}{2400} \left[ \frac{t^4}{4} \right]_{6}^{8} = \frac{1}{4 \times 2400} \left[ 8^4 - 6^4 \right] = \frac{2800}{4096} = \frac{175}{256}
\]

(d) The probability that the television breaks down within 2 years or after 6 years is given by adding the answers to parts (a), (b) and (c). We have

\[
P(0 \leq T \leq 2 \text{ or } 6 \leq T \leq 8) = P(0 \leq T \leq 2) + P(6 \leq T \leq 8)
\]

\[
= \frac{1}{4096} + \frac{15}{4096} + \frac{2800}{4096} + \frac{2816}{4096} = \frac{11}{16}
\]

3. (a) The mean time for completion \( \mu = E(T) = \int_{0}^{\infty} tf(t) \, dt \) with \( f(t) = \frac{t}{48} \left( 1 - \frac{t}{24} \right) \):

\[
\mu = E(T) = \int_{0}^{12} t \frac{t}{48} \left( 1 - \frac{t}{24} \right) \, dt
\]

\[
= \frac{1}{48} \int_{0}^{12} t^2 \left( \frac{24 - t}{24} \right) \, dt
\]

\[
= \frac{1}{48 \times 24} \left[ 24t^2 - t^3 \right]_{0}^{12} = \frac{1}{1152} \left[ 8 \left( 12 \right)^3 - \frac{12^5}{4} \right] = \frac{8640}{1152} = 7 \frac{1}{2}
\]

The mean time for completion is \( 7 \frac{1}{2} \) hours.

(b) We evaluate \( E(T^2) = \int_{0}^{12} t^2 f(t) \, dt \) with \( f(t) = \frac{t}{48} \left( 1 - \frac{t}{24} \right) \) and limits 0, 12:

\[
E(T^2) = \int_{0}^{12} t^2 \frac{t}{48} \left( 1 - \frac{t}{24} \right) \, dt
\]

\[
= \frac{1}{48} \int_{0}^{12} t^3 \left( \frac{24 - t}{24} \right) \, dt
\]

\[
= \frac{1}{48 \times 24} \left[ 24t^3 - t^4 \right]_{0}^{12} = \frac{1}{1152} \left[ 6 \left( 12 \right)^4 - \frac{12^5}{5} \right] = \frac{64}{5}
\]

(c) The variance is given by the formula \( \sigma^2 = E(T^2) - \mu^2 \). By substituting our answers from parts (a) and (b), \( \mu = 7 \frac{1}{2} \) and \( E(T^2) = \frac{64}{5} \) into this formula we have
\[
\sigma^2 = E(T^2) - \mu^2 = 64 \frac{4}{5} \left( \frac{7}{2} \right)^2 = 8 \frac{11}{20}
\]

13. (i) The mean \( \mu = E(T) \) which means that

\[
\mu = E(T) = \int_a^b t \left( \frac{1}{b-a} \right) dt
\]

\[
= \frac{1}{b-a} \int_a^b t \ dt \quad \text{[Taking out} \ \frac{1}{b-a} \text{]} \\
= \frac{1}{b-a} \left[ \frac{t^2}{2} \right]_a^b \quad \text{[Integrating]}
\]

\[
= \frac{1}{2(b-a)} \left[ b^2 - a^2 \right] = \frac{1}{2(b-a)} \left[ (b-a)(b+a) \right] = \frac{b+a}{2}
\]

This is our required result.

(ii) To find the standard deviation we first determine the variance \( \sigma^2 \) and then take the square root:

\[
\sigma^2 = E(T^2) - \mu^2
\]

\[
= E(T^2) - \left( \frac{a+b}{2} \right)^2 \quad \text{[Because by part (i)} \ \mu = \frac{a+b}{2} \text{]} 
\]

We need to determine \( E(T^2) \).

\[
E(T^2) = \int_a^b t^2 \left( \frac{1}{b-a} \right) dt
\]

\[
= \frac{1}{b-a} \int_a^b t^2 \ dt \quad \text{[Taking out} \ \frac{1}{b-a} \text{]} \\
= \frac{1}{b-a} \left[ \frac{t^3}{3} \right]_a^b \quad \text{[Integrating]}
\]

\[
= \frac{1}{3(b-a)} \left[ b^3 - a^3 \right] = \frac{1}{3(b-a)} \left[ (b-a)(b^2 + ab + a^2) \right] = \frac{b^2 + ab + a^2}{3} \quad \text{[Cancelling} \ b-a \text{]}
\]

Substituting \( E(T^2) = \frac{b^2 + ab + a^2}{3} \) into \( \sigma^2 = E(T^2) - \left( \frac{a+b}{2} \right)^2 \) gives
\[ \sigma^2 = \frac{b^2 + ab + a^2}{3} - \left( \frac{a + b}{2} \right)^2 \]

\[ = \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4} \]

\[ = \frac{4b^2 + 4ab + 4a^2 - 3a^2 - 6ab - 3b^2}{12} \]

\[ = \frac{b^2 - 2ab + a^2}{12} = \frac{(b - a)^2}{12} \]

Taking the square root of this gives

\[ \sigma = \sqrt{\frac{(b-a)^2}{12}} = \frac{b-a}{\sqrt{12}}. \]