Complete Solutions to Examination Questions 14

1. The characteristic equation is

\[ m^2 + 2m + 2 = 0 \]

Solving this quadratic equation by the formula method \( m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \) with \( a = 1, \ b = 2 \) and \( c = 2 \) gives

\[
m = \frac{-2 \pm \sqrt{2^2 - (4 \times 1 \times 2)}}{2 \times 1} = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm j
\]

The general solution is given by

\[ y = e^{-x} [A \cos(x) + B \sin(x)] \quad (*)\]

Substituting the given initial condition \( y = 1 \) at \( x = 0 \) into (*):

\[
e^{-0} [A \cos(0) + B \sin(0)] = A = 1
\]

To use the other initial condition we need to differentiate (*):

\[
\frac{dy}{dx} = -e^{-x} [A \cos(x) + B \sin(x)] + e^{-x} [-A \sin(x) + B \cos(x)]
\]

Substituting the other initial condition \( \frac{dy}{dx} = 0 \) at \( x = 0 \) into this result:

\[
e^{-0} [-A \cos(0) - B \sin(0) - A \sin(0) + B \cos(0)] = -A + B = 0
\]

From above we have \( A = 1 \) therefore \( B = 1 \). Our general solution is found by substituting these values \( A = 1 \) and \( B = 1 \) into (*):

\[ y = e^{-x} [\cos(x) + \sin(x)] \]

2. (i) The characteristic equation is \( m^2 + 16 = 0 \). Solving this gives

\[ m^2 = -16 \]

\[ m = \pm 4j = 0 \pm j4 \]

Since we have complex roots

\[ y = A \cos(4x) + B \sin(4x) \quad (*)\]

We are given the initial conditions \( y(0) = 3 \) and \( y'(0) = -2 \). Substituting the first of these conditions \( y(0) = 3 \) which means that when \( x = 0 \), \( y = 3 \) into (*):

\[ A \cos(4 \times 0) + B \sin(4 \times 0) = 3 \quad \text{gives} \ A = 3 \]
Next we apply the second condition \( y'(0) = -2 \), which means that when \( x = 0 \), \( y' = -2 \).

Differentiating (*) gives

\[
y' = -4A \sin(4x) + 4B \cos(4x)
\]

Because \( \cos'(kx) = -k \sin(kx) \)

and \( \sin'(kx) = k \cos(kx) \)

Substituting \( x = 0 \), \( y' = -2 \) into the above yields

\[
-4A \sin(4\times0) + 4B \cos(4\times0) = -2
\]

\[
4B = -2 \implies B = -\frac{2}{4} = -\frac{1}{2}
\]

Our particular solution is found by substituting \( A = 3 \) and \( B = -\frac{1}{2} \) into (*):

\[
y = A \cos(4x) + B \sin(4x) = 3 \cos(4x) - \frac{1}{2} \sin(4x)
\]

(ii) For solving the given differential equation \( \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - 15y = 2e^{4x} \) we first find the homogeneous solution:

\[
\frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - 15y = 0
\]

The characteristic equation is

\[
m^2 + 2m - 15 = 0
\]

\[
(m - 3)(m + 5) = 0 \quad \text{[Factorizing]}
\]

\[
m_1 = 3, \quad m_2 = -5
\]

Our complementary function is \( y_c = Ae^{3x} + Be^{-5x} \). What is the trail function for the particular integral in this case?

\( Y = Ce^{4x} \). Differentiating this gives

\[
Y' = 4Ce^{4x} \quad \text{[Because \( (e^{kx})' = ke^{kx} \)]}
\]

\[
Y'' = 16Ce^{4x} \quad \text{[Because \( (e^{kx})' = ke^{kx} \)]}
\]

Substituting these into the given differential equation \( \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - 15y = 2e^{4x} \) yields

\[
16Ce^{4x} + 2(4Ce^{4x}) - 15Ce^{4x} = 2e^{4x}
\]

\[
(16C + 8C - 15C)e^{4x} = 2e^{4x}
\]

\[
9C = 2 \quad \text{implies that} \quad C = \frac{2}{9}
\]

Hence the particular integral is \( Y = Ce^{4x} = \frac{2}{9}e^{4x} \).

Our general solution to the differential equation is
\[ y = y_e + Y = Ae^{3x} + Be^{-5x} + \frac{2}{9}e^{4x}. \]

3. (a) We need to solve the given differential equation \( \frac{d^2s}{dt^2} + 25s = 0 \). The characteristic equation is

\[ m^2 + 25 = 0 \quad \text{gives} \quad m = \pm j5 \]

The general solution is

\[ s = A \cos(5t) + B \sin(5t) \quad \therefore (\ddagger) \]

Substituting the given initial condition when \( t = 0, \ s = 2 \) we have

\[ A \cos(5\times0) + B \sin(5\times0) = 2 \quad \text{gives} \quad A = 2 \]

The other initial condition is \( \frac{ds}{dt} = 5 \) when \( t = 0 \). Differentiating \((\ddagger)\) yields

\[ \frac{ds}{dt} = -5A \sin(5t) + 5B \cos(5t) \quad \text{Because} \quad \left[ \cos(kt) \right]' = -k \sin(kt) \quad \text{and} \quad \left[ \sin(kt) \right]' = k \cos(kt) \]

Substituting the other initial condition:

\[ -5A \sin(5\times0) + 5B \cos(5\times0) = 5 \quad \text{gives} \quad 5B = 5 \quad \Rightarrow \quad B = 1 \]

Putting \( A = 2 \) and \( B = 1 \) into \((\ddagger)\) gives

\[ s = 2 \cos(5t) + \sin(5t) \]

(b) We need to find \( s \) at \( t = \frac{\pi}{4} \):

\[ s = 2 \cos \left( \frac{5\pi}{4} \right) + \sin \left( \frac{5\pi}{4} \right) \]

\[ = 2 \left( -\frac{1}{\sqrt{2}} \right) + \left( -\frac{1}{\sqrt{2}} \right) = -\frac{1}{\sqrt{2}} [2 + 1] = -\frac{3}{\sqrt{2}} \]

(c) The spring is initially at rest means that the acceleration \( \frac{d^2y}{dt^2} = 0 \). Using \( \frac{d^2s}{dt^2} + 25s = 0 \)

gives that \( s = 0 \).

4. The characteristic equation of \( \frac{d^2y}{dx^2} + y = 0.001x^2 \) is

\[ m^2 + 1 = 0 \quad \text{implies} \quad m = \pm j1 \]

The complementary function is given by

\[ y_c = A \cos(x) + B \sin(x) \]

Since \( f(x) = 0.001x^2 \) we trail the particular integral

\[ Y = Cx^2 + Dx + E \]

Differentiating this gives
\[ \frac{d^2Y}{dx^2} = 2C \]
\[ \frac{dY}{dx} = 2C + D \]

By substituting these into the given differential equation \( \frac{d^2y}{dx^2} + y = 0.001x^2 \) we can find the values of \( C, D, \) and \( E \). We have

\[ 2C + Cx^2 + Dx + E = 0.001x^2 \]
\[ Cx^2 + Dx + (2C + E) = 0.001x^2 \] (†)

Equating the \( x^2 \) coefficients of (†):
\[ C = 0.001 \]

Equating the \( x \) coefficients of (†):
\[ D = 0 \]

Equating the constant coefficients of (†):
\[ 2C + E = 0 \]
\[ 2(0.001) + E = 0 \] (Because \( C = 0.001 \))
\[ E = -0.002 \]

Thus the particular integral is found by substituting these values \( C = 0.001, D = 0 \) and \( E = -0.002 \) into \( Y = Cx^2 + Dx + E \) which gives
\[ Y = 0.001x^2 - 0.002 \]

Hence our general solution is given by
\[ y = Y + \dot{Y} = A\cos(x) + B\sin(x) + 0.001x^2 - 0.002 \] (*)

We can find the values of \( A \) and \( B \) by using the given initial conditions
\[ y(0) = 0, \quad \dot{y}(0) = 1.5 \]
\[ y(0) = 0 \] means that when \( x = 0, \ y = 0 \). Substituting this into (*) gives
\[ A\cos(0) + B\sin(0) + 0.001(0)^2 - 0.002 = A - 0.002 = 0 \]

Hence \( A = 0.002 \). To use the other initial condition \( \dot{y}(0) = 1.5 \) which means that when \( x = 0, \ \dot{y} = 1.5 \), so we need to differentiate (*):
\[ \dot{y} = -A\sin(x) + B\cos(x) + 0.002x \]

Substituting \( x = 0 \) and \( \dot{y} = 1.5 \) into this yields
\[ -A\sin(0) + B\cos(0) + 0.002(0) = B = 1.5 \]

Our particular solution is found by substituting \( A = 0.002 \) and \( B = 1.5 \) into (*):
\[ y = 0.002\cos(x) + 1.5\sin(x) + 0.001x^2 - 0.002 \]
\[ = 1.5\sin(x) + 0.001[2\cos(x) + x^2 - 2] \]

5. The characteristic equation of \( \frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = -4x \) is
Solving this quadratic equation:
\[ m^2 + m - 2 = 0 \]
\[ (m - 1)(m + 2) = 0 \]
gives \( m_1 = 1, m_2 = -2 \)

Our complementary function is \( y_c = Ae^t + Be^{-2t} \). What is our trial function in this case?

Since we have \( f(x) = -4x \) therefore \( Y = Cx + D \). Differentiating this:
\[
\begin{align*}
Y &= Cx + D \\
Y' &= C \\
Y'' &= 0
\end{align*}
\]

Substituting these results into the given differential equation
\[
\frac{d^2y}{dx^2} + 2y = -4x
\]
yields
\[
0 + C - 2(Cx + D) = -4x \\
C - 2Cx - 2D = -4x
\]

Equating coefficients of \( x \):
\[ -2C = -4 \quad \text{implies that} \quad C = 2 \]

Equating constants:
\[ C - 2D = 0 \]
\[ 2 - 2D = 0 \quad \text{implies that} \quad D = \frac{2}{2} = 1 \]

Our particular integral is \( Y = Cx + D = 2x + 3 \). This means that our general solution is
\[
y = y_c + Y = Ae^t + Be^{-2t} + 2x + 1 \quad (*)
\]

We need to find the particular solution which satisfies the given initial conditions \( y(0) = 4, \)
\( y'(0) = 5 \). What does \( y(0) = 4 \) mean?

When \( x = 0 \), \( y = 4 \). Substituting these into (*) yields
\[
Ae^0 + Be^{-2(0)} + (2 \times 0) + 1 = A + B + 0 + 1 = 4 \\
A + B = 4 - 1 = 3
\]

To use the other initial condition we need to differentiate (*):
\[
y' = Ae^t - 2Be^{-2t} + 2
\]

Applying the initial condition \( y'(0) = 5 \) we have
\[
Ae^0 - 2Be^{-2(0)} + 2 = A - 2B + 2 = 5 \\
A - 2B = 5 - 2 = 3
\]

Solving the simultaneous equations
\[
\begin{align*}
A + B &= 3 \\
A - 2B &= 3
\end{align*}
\]
implies that \( A = 3, \ B = 0 \)

Our particular solution is given by substituting these values of \( A = 3 \) and \( B = 0 \) into (*):
\[
y = Ae^t + Be^{-2t} + 2x + 1 \\
= 3e^t + 0e^{-2t} + 2x + 1 = 3e^t + 2x + 1
\]

6. (a) The characteristic equation of \( 2y'' + 5y' + 3y = 0 \) is
\[ 2m^2 + 5m + 3 = 0 \]
\[ (2m+3)(m+1) = 0 \] implies that \( m_1 = -\frac{3}{2}, \ m_2 = -1 \)

Our general solution is
\[ y = Ae^{\frac{3}{2}x} + Be^{-x} \] (*)

Substituting the given initial condition \( y(0) = 3 \) [when \( x = 0, \ y = 3 \) ] into (*):
\[ Ae^{\frac{3}{2}0} + Be^{0} = A + B = 3 \]

We need to differentiate (*) to use the other initial condition:
\[ \frac{dy}{dx} = \frac{3}{2}Ae^{\frac{3}{2}x} - Be^{-x} \quad \text{Using} \quad (e^{ix})' = ke^{ix} \]

Applying the other initial condition \( y'(0) = -4 \) which means that when \( x = 0, \ y' = -4 \):
\[ -\frac{3}{2}Ae^{\frac{3}{2}0} - Be^{0} = -\frac{3}{2}A - B = -4 \]

We need to solve the simultaneous equations
\[ \begin{align*}
A + B &= 3 \\
-\frac{3}{2}A - B &= -4
\end{align*} \]
gives \( A = 2 \) and \( B = 1 \)

The particular solution is determined by substituting \( A = 2 \) and \( B = 1 \) into (*):
\[ y = Ae^{\frac{3}{2}x} + Be^{-x} = 2e^{\frac{3}{2}x} + e^{-x} \]

The solution is \( y = 2e^{\frac{3}{2}x} + e^{-x} \).

(b) First we find the characteristic equation of \( y'' - y' = \sin(2x) \):
\[ m^2 - m = 0 \]
\[ m(m-1) = 0 \quad m_1 = 0, \ m_2 = 1 \]

The complementary function is given by
\[ y_c = Ae^0 + Be^x = A + Be^x \quad \text{[Because} \ e^0 = 1] \]

Our trial function is \( Y = C \cos(2x) + D \sin(2x) \). We need to differentiate this in order to find the values of \( C \) and \( D \).
\[ Y = C \cos(2x) + D \sin(2x) \]
\[ \frac{dY}{dx} = -2C \sin(2x) + 2D \cos(2x) \]
\[ \frac{d^2Y}{dx^2} = -4C \cos(2x) - 4D \sin(2x) \]

Putting this into the given differential equation \( y'' - y' = \sin(2x) \):
\[ -4C \cos(2x) - 4D \sin(2x) - [ -2C \sin(2x) + 2D \cos(2x) ] = \sin(2x) \]
\[ (-4C - 2D) \cos(2x) + (2C - 4D) \sin(2x) = \sin(2x) \]

Equating coefficients of \( \cos(2x) \):
Equating coefficients of \( \sin(2x) \):

\[
-4C - 2D = 0
\]

\[
2C - 4D = 1
\]

Solving these simultaneous equations:

\[
\begin{align*}
-4C - 2D &= 0 \\
2C - 4D &= 1
\end{align*}
\]

\[
\Rightarrow C = \frac{1}{10}, \quad D = -\frac{1}{5}
\]

The particular integral is

\[
Y = C \cos(2x) + D \sin(2x)
\]

\[
= \frac{1}{10} \cos(2x) - \frac{1}{5} \sin(2x) = \frac{1}{10} \left[ \cos(2x) - 2 \sin(2x) \right]
\]

The general solution is given by

\[
y = y_c + Y = A + Be^x + \frac{1}{10} \left[ \cos(2x) - 2 \sin(2x) \right]
\]

7. (a) We can test the function \( y = \frac{1}{2} x \sin(x) \) is a solution of the given differential equation by differentiating this and then substituting the results into the differential equation.

\[
y = \frac{1}{2} x \sin(x)
\]

\[
\frac{dy}{dx} = \frac{1}{2} \left[ \sin(x) + x \cos(x) \right]
\]

Using the product rule

\[
\frac{d^2y}{dx^2} = \frac{1}{2} \left[ \cos(x) + \cos(x) - x \sin(x) \right] = \frac{1}{2} \left[ 2 \cos(x) - x \sin(x) \right]
\]

Substituting these results into the LHS of \( y'' + y = \sin(x) \) gives

\[
y'' + y = \frac{1}{2} \left[ 2 \cos(x) - x \sin(x) \right] + \frac{1}{2} x \sin(x)
\]

\[
= \cos(x)
\]

Thus \( y = \frac{1}{2} x \sin(x) \) is not a solution of the given differential equation \( y'' + y = \sin(x) \).

(b) This time we test \( y = -\frac{1}{2} x \cos(x) \). We have

\[
y = -\frac{1}{2} x \cos(x)
\]

\[
\frac{dy}{dx} = -\frac{1}{2} \left[ \cos(x) - x \sin(x) \right] \quad \text{[By product rule]}
\]

\[
\frac{d^2y}{dx^2} = -\frac{1}{2} \left[ -\sin(x) - (\sin(x) + x \cos(x)) \right]
\]

\[
= -\frac{1}{2} \left[ -2 \sin(x) - x \cos(x) \right] = \frac{1}{2} \left[ 2 \sin(x) + x \cos(x) \right]
\]

Substituting this into the LHS of \( y'' + y = \sin(x) \) gives
\[ y'' + y = \frac{1}{2} [2 \sin(x) + x \cos(x)] - \frac{1}{2} x \cos(x) \]
\[ = \sin(x) \]

This \( y = -\frac{1}{2} x \cos(x) \) is a solution of the given differential equation.
8. We need to solve \( \frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 2y = e^{3x} \). We first find the complementary function:

\[
m^2 + m - 2 = 0
\]

\((m-1)(m+2) = 0\) gives \( m_1 = 1, \ m_2 = -2 \)

Our complementary function is equal to

\[
y_c = Ae^x + Be^{-2x}
\]

Because \( f(x) = e^{3x} \) therefore our trial function is \( Y = Ce^{3x} \). Differentiating this gives

\[
\frac{dY}{dx} = 3Ce^{3x} \quad \text{Because} \quad \left( e^{3x} \right)' = ke^{3x}
\]

\[
\frac{d^2Y}{dx^2} = 9Ce^{3x} \quad \text{Because} \quad \left( e^{3x} \right)'' = ke^{3x}
\]

Substituting these into the given differential equation \( \frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 2y = e^{3x} \):

\[
9Ce^{3x} + 3Ce^{3x} - 2Ce^{3x} = e^{3x}
\]

\[
10Ce^{3x} = e^{3x} \implies C = \frac{1}{10}
\]

The particular integral is \( Y = Ce^{3x} = \frac{1}{10}e^{3x} \). The general solution is given by

\[
y = y_c + Y = Ae^x + Be^{-2x} + \frac{1}{10}e^{3x}
\]

The solution to the given differential equation is \( Ae^x + Be^{-2x} + \frac{1}{10}e^{3x} \).

9. The characteristic equation is

\[
m^2 + 1 = 0 \implies m = \pm \sqrt{-1} = \pm j
\]

Our complementary function \( y_c \) is

\[
y_c = A\cos(x) + B\sin(x)
\]

The trial function in this case is

\[
Y = x\left[ C\cos(x) + D\sin(x) \right]
\]

Differentiating this function by using the product rule we have

\[
\frac{dY}{dx} = \left[ C\cos(x) + D\sin(x) \right] + x \left[ -C\sin(x) + D\cos(x) \right]
\]

\[
= \left[ C + Dx \right]\cos(x) + \left[ D - Cx \right]\sin(x)
\]

\[
\frac{d^2Y}{dx^2} = D\cos(x) + \left[ C + Dx \right]\left[ -\sin(x) \right] + \left[ -C \right]\sin(x) + \left[ D - Cx \right]\cos(x)
\]

\[
= \left[ 2D - Cx \right]\cos(x) - \left[ 2C + Dx \right]\sin(x)
\]

Substituting this into the given differential equation \( y'' + y = \cos x \):

\[
2D\cos(x) - 2C\sin(x) + x\left[ C\cos(x) + D\sin(x) \right] = \cos(x)
\]

\[
2\cos(x) - 2C\sin(x) = \cos(x)
\]

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From the last equation we have $D = \frac{1}{2}$ and $C = 0$. This means our particular integral is

$$Y = x\left[ C\cos(x) + D\sin(x) \right]$$

$$= x\left[ (0)\cos(x) + \frac{1}{2}\sin(x) \right] = \frac{1}{2}x\sin(x)$$

Thus our general solution is

$$y = y_c + Y = A\cos(x) + B\sin(x) + \frac{1}{2}x\sin(x) \quad (*)$$

We can find a particular solution because we have been given initial conditions. Substituting the initial condition $y'(0) = 0$ which means that when $x = 0$ then $y = 0$:

$$A\cos(0) + B\sin(0) + \frac{1}{2}(0)\sin(0) = A = 0$$

The other initial condition is $y'(0) = 5/2$ means that when $x = 0$, $y' = 5/2$. We need to differentiate (*) in order to use this condition:

$$y' = -A\sin(x) + B\cos(x) + \frac{1}{2}\left[ \sin(x) + x\cos(x) \right]$$

Using the product rule

Substituting $x = 0$ and $y' = 5/2$ into this yields:

$$-A\sin(0) + B\cos(0) + \frac{1}{2}\left[ \sin(0) + 0\cos(0) \right] = B = \frac{5}{2}$$

Hence our particular solution is given by putting $A = 0$ and $B = \frac{5}{2}$ into (*):

$$y = A\cos(x) + B\sin(x) + \frac{1}{2}x\sin(x)$$

$$= 0\cos(x) + \frac{5}{2}\sin(x) + \frac{1}{2}x\sin(x)$$

$$= \frac{1}{2}\left[ 5\sin(x) + x\sin(x) \right]$$

Our solution is $y = \frac{1}{2}\left[ 5\sin(x) + x\sin(x) \right]$.

10. The characteristic equation is given by

$$m^2 + 9 = 0 \quad \text{gives} \quad m = \pm 3i$$

Our complementary function is

$$y_c = A\cos(3t) + B\sin(3t)$$

Our trial function is

$$Y = C\cos^2(3t) + D\cos(3t) + E + Ft\cos(3t) + Gt\sin(3t)$$

We need to find the values of the unknowns $C$, $D$, $E$, $F$ and $G$. How?

By differentiating twice and substituting into the given differential equation:
\[ \frac{dY}{dx} = 2Ct + D + F\left[ (1) \cos(3t) - 3t \sin(3t) \right] + G\left[ (1) \sin(3t) + 3t \cos(3t) \right] \]
\[ = 2Ct + D + F \cos(3t) - 3tF \sin(3t) + G \sin(3t) + 3Gt \cos(3t) \]

\[ \frac{d^2Y}{dx^2} = 2C - 3F \sin(3t) - 3F\left[ (1) \sin(3t) + 3t \cos(3t) \right] + 3G \cos(3t) + 3G\left[ (1) \cos(3t) - 3t \sin(3t) \right] \]
\[ = 2C - 3F \sin(3t) - 3F \sin(3t) - 9F \cos(3t) + 3G \cos(3t) + 3G \cos(3t) - 9Gt \sin(3t) \]
\[ = 2C - 6F \sin(3t) - 9F \cos(3t) + 6G \cos(3t) - 9Gt \sin(3t) \]

Substituting these results into the differential equation yields
\[ \frac{d^2y}{dt^2} + 9y = 9t^2 - 12 \cos(3t) \]
\[ 2C - 6F \sin(3t) - 9F \cos(3t) + 6G \cos(3t) - 9Gt \sin(3t) \]
\[ + 9\left( Ct^2 +Dt + E + F1 \cos(3t) + Gt \sin(3t) \right) = 9t^2 - 12 \cos(3t) \]
\[ 2C - 6F \sin(3t) - 9F \cos(3t) + 6G \cos(3t) - 9Gt \sin(3t) \]
\[ + 9Ct^2 + 9Dt + 9E + 9F1 \cos(3t) + 9Gt \sin(3t) = 9t^2 - 12 \cos(3t) \]
\[ 2C - 6F \sin(3t) + 6G \cos(3t) + 9Ct^2 + 9Dt + 9E = 9t^2 - 12 \cos(3t) \] (†)

Equating coefficients of \( t^2 \) in (†):
\[ 9C = 9 \quad \text{gives} \quad C = 1 \]

Equating coefficients of \( t \) in (†):
\[ 9D = 0 \quad \text{gives} \quad D = 0 \]

Equating coefficients of constants in (†):
\[ 2C + 9E = 0 \]

From above we have \( C = 1 \) therefore \( 2C + 9E = 2 + 9E = 0 \) implies that \( E = -\frac{2}{9} \).

Equating coefficients of \( \cos(3t) \) in (†):
\[ 6G = -12 \quad \text{gives} \quad G = -2 \]

Equating coefficients of \( \sin(3t) \) in (†):
\[ -6F = 0 \quad \text{gives} \quad F = 0 \]

We have \( C = 1 \), \( D = 0 \), \( E = -\frac{2}{9} \), \( F = 0 \) and \( G = -2 \). Thus our particular integral is

\[ Y = Ct^2 + Dt + E + F1 \cos(3t) + Gt \sin(3t) \]
\[ = t^2 + (0)t - \frac{2}{9}(0)t \cos(3t) - 2t \sin(3t) \]
\[ = t^2 - \frac{2}{9} - 2t \sin(3t) \]

Our general solution is
\[ y = y_c + Y = A \cos(3t) + B \sin(3t) + t^2 - \frac{2}{9} - 2t \sin(3t) \]

11. (a) We need to solve \( \dot{x} = -9x \) which has the characteristic equation given by
\[ m^2 = -9 \quad \text{implies} \quad m = \pm \sqrt{-9} = \pm 3i \]

Our general solution is
\[ x = A \cos(3t) + B \sin(3t) \]  \quad (1)

Substituting the given initial condition \( x = -1 \) when \( t = 0 \) into (1):

\[ A \cos(3 	imes 0) + B \sin(3 	imes 0) = A = -1 \]

To use the other initial condition \( \dot{x} = 3 \) when \( t = 0 \) we need to differentiate (1):

\[ \dot{x} = -3A \sin(3t) + 3B \cos(3t) \]

Substituting \( t = 0, \dot{x} = 3 \) into this:

\[ -3A \sin(3 	imes 0) + 3B \cos(3 	imes 0) = 3B = 3 \quad \text{gives} \ B = 1 \]

Our particular solution is found by putting \( A = -1 \) and \( B = 1 \) into (1):

\[ x = A \cos(3t) + B \sin(3t) = -\cos(3t) + \sin(3t) \]

We need to place this into amplitude-phase form \( R \sin(\omega t + \phi) \). In general

\[ a \cos(\theta) + b \sin(\theta) = R \sin(\theta + \alpha) \quad \text{where} \quad R = \sqrt{a^2 + b^2} \quad \text{and} \quad \alpha = \tan^{-1}\left(\frac{a}{b}\right) \]

For \( x = -\cos(3t) + \sin(3t) \) we have \( a = -1 \) and \( b = 1 \) therefore \( R = \sqrt{(-1)^2 + 1^2} = \sqrt{2} \) and

\[ \alpha = \tan^{-1}\left(-\frac{1}{1}\right) = -\frac{\pi}{4} \]

We have

\[ x = -\cos(3t) + \sin(3t) = \sqrt{2} \sin\left(3t - \frac{\pi}{4}\right) \]

The amplitude is \( R = \sqrt{2} \) and period \( \frac{2\pi}{3} \).

To sketch the graph of \( x = \sqrt{2} \sin\left(3t - \frac{\pi}{4}\right) = \sqrt{2} \sin\left[3\left(t - \frac{\pi}{12}\right)\right] \) is the sine graph with

amplitude of \( \sqrt{2} \) and covering 3 cycles between 0 to \( 2\pi \) and shifted to the right by \( \frac{\pi}{12} \) rad.
(b) We first find the homogeneous solution of
\[ \frac{d^2x}{dt^2} - 2 \frac{dx}{dt} + 10x = 0 \]
which means we have zero on the RHS:
\[ \frac{d^2x}{dt^2} - 2 \frac{dx}{dt} + 10x = 0 \]
The characteristic equation is given by
\[ m^2 - 2m + 10 = 0 \]
To solve this quadratic equation we need to use the formula:
\[
m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - (4 \times 10)}}{2} = \frac{2 \pm \sqrt{-36}}{2} = \frac{2 \pm j6}{2} = 1 \pm j3
\]
Our complementary function is given by
\[ x_c = e^{(1+j3)t} \sqrt{ \cos(3t) + j \sin(3t)} \]
Our trial function is \[ X = Ct + D \]. Differentiating this gives
\[
X = Ct + D \quad X' = C \quad X'' = 0
\]
Substituting these into the given differential equation
\[ \frac{d^2x}{dt^2} - 2 \frac{dx}{dt} + 10x = 20t + 6 \]:
\[
0 - 2C + 10(Ct + D) = 10Ct - 2C + 10D = 20t + 6
\]
Equating coefficients of \( t \):
\[ 10C = 20 \quad \text{gives} \quad C = 2 \]
Equating constants:
\[ -2C + 10D = -2(2) + 10D = 6 \quad \text{implies} \quad D = 1 \]
The particular integral is equal to \( X = 2t + 1 \). Hence our solution is given by
\[ x = x_c + X = e^{(1+j3)t} \sqrt{ \cos(3t) + j \sin(3t)} + 2t + 1 \]