Solutions to Miscellaneous Exercise 7

1.

![Diagram](image)

The volume = 100 m$^3$. So

\[ 2xy = 100 \]

\[ y = \frac{100}{2x} = \frac{50}{x} \quad (*) \]

The surface area, \(A\), consists of the bottom part, 2 sides, the front and the back. Thus

\[ A = 2x + (2y + 2y) + (xy + xy) = 2x + 4y + 2xy \]

\[ \equiv 2x + 4\left( \frac{50}{x} \right) + 2x\left( \frac{50}{x} \right) \]

\[ A = 2x + \frac{200}{x} + 100 = 2x + 200x^{-1} + 100 \]

For stationary points:

\[ \frac{dA}{dx} = 2 - 200x^{-2} = 0 \]

\[ 2 = 200x^{-2} = \frac{200}{x^2} \]

\[ x^2 = \frac{200}{2} = 100 \]

How can we find \(x\)?

Take the square root of both sides: \(x = \sqrt{100} = +10, -10\)

Since \(x\) is length it cannot be \(-10\). So \(x = 10\) m.

To check that \(x = 10\) m gives minimum surface area we have to differentiate again:

\[ \frac{d^2A}{dx^2} = 400x^{-3} = \frac{400}{x^3} > 0 \] (because \(x > 0\))

By (7.3), \(x = 10\) m gives minimum surface area. What is the value of \(y\)?

We can find \(y\) from (*) by substituting \(x = 10\): \(y = 50/10 = 5\) m.

Thus \(x = 10\) m, \(y = 5\) m gives minimum surface area.

2. Let \(x\) and \(y\) represent the dimensions as shown below:

![Diagram](image)

The perimeter of the cross section is \(4x\) so

\[ 4x + y = 2 \]

\[ y = 2 - 4x \quad (*) \]

The volume, \(v\), of the parcel is given by

\[ v = x^2 y = x^2 (2 - 4x) = 2x^2 - 4x^3 \]

(7.3) \(A' = 0, \ A'' > 0 \) minimum
To find stationary points
\[
\frac{dv}{dx} = 4x - 12x^2 = 0, \quad 4x(1 - 3x) = 0 \text{ which gives } x = 0, \quad x = 1/3
\]
x = 0 m is not a feasible solution, why not?
If x = 0 m then we will not have a parcel. For x = 1/3 m, we can use the second derivative test:
\[
\frac{d^2v}{dx^2} = 4 - 24x
\]
Substituting \( x = \frac{1}{3} \) gives \( \frac{d^2v}{dx^2} = 4 - 8 = -4 < 0 \). By (7.2), \( x = \frac{1}{3} \) m gives maximum volume. To find \( y \) we substitute \( x = 1/3 \) into (*):
\[
y = 2 - \frac{4}{3} = \frac{2}{3}
\]
Hence \( x = 1/3 \) m, \( y = 2/3 \) m gives maximum volume.

3. Similar to solution 2. Let \( L \) represent the sum of length and girth.
\[
4x + y = L
\]
\[
y = L - 4x \quad \text{(†)}
\]
The volume \( v \) is given by
\[
v = x^2y = x^2(L - 4x)
\]
\[
v = Lx^2 - 4x^3
\]
\[
\frac{dv}{dx} = 2Lx - 12x^2 = 0, \quad 2x(L - 6x) = 0 \text{ gives } x = 0, \quad x = L/6
\]
As before \( x = L/6 \). Differentiating again:
\[
\frac{d^2v}{dx^2} = 2L - 24x
\]
At \( x = \frac{L}{6} \), \( \frac{d^2v}{dx^2} = 2L - 24\left(\frac{L}{6}\right) = 2L - 4L = -2L < 0 \) [Negative]
By (7.2), \( x = L/6 \) gives maximum volume. Substituting \( x = L/6 \) into (†):
\[
y = L - \frac{4L}{6} = \frac{2L}{6} = 2x \quad \text{. Hence the length is twice the side of the square.}
\]

4. We have
\[
w = -36x^2 + 50x
\]
\[
\frac{dw}{dx} = -72x + 50 = 0, \quad 72x = 50 \text{ gives } x = \frac{50}{72} = \frac{25}{36} \text{ m}
\]
\[
\frac{d^2w}{dx^2} = -72 < 0 \quad \text{[Negative]}
\]
By (7.2) at \( x = 25/36 \) m the loading is maximum.

5.
\[
y = \frac{1}{12 \times 10^2}\left(x^4 - 14x^3 + 36x^2\right) \quad \text{(*)}
\]
\[
\frac{dy}{dx} = \frac{1}{12 \times 10^3}\left(4x^3 - 42x^2 + 72x\right)
\]
\[
y' = 0, \quad y'' < 0 \text{ maximum}
\]
For stationary points we have
\[
\frac{2x}{12 \times 10^3} (2x^2 - 21x + 36) = 0
\]
\[
x = 0 \quad \text{or} \quad 2x^2 - 21x + 36 = 0
\]
We solve the quadratic by putting \(a = 2\), \(b = -21\) and \(c = 36\) into (1.16), which gives:
\[
x = \frac{21 \pm \sqrt{(21)^2 - (4 \times 2 \times 36)}}{4} = 2.16 \text{ m or } 8.34 \text{ m}
\]
\(x\) cannot be 8.34 m because the beam is only 3m long so \(x = 2.16 \text{ m}\). To check the nature of the stationary point we need to differentiate again:
\[
\frac{dy}{dx} = \frac{1}{12 \times 10^3} (4x^3 - 42x^2 + 72x)
\]
\[
\frac{d^2y}{dx^2} = \frac{1}{12 \times 10^3} (12x^2 - 84x + 72) = \frac{12}{12 \times 10^3} (x^2 - 7x + 6) = (x^2 - 7x + 6) \times 10^{-3}
\]
At \(x = 2.16\), \(\frac{d^2y}{dx^2} = (2.16^2 - (7 \times 2.16) + 6) \times 10^{-3} = -4.45 \times 10^{-3} < 0\) [Negative]

By (7.2), \(x = 2.16 \text{ m}\) gives maximum deflection. To find the maximum deflection we substitute \(x = 2.16\) into (*):
\[
y = \frac{1}{12 \times 10^3} \left[ 2.16^4 - (14 \times 2.16^3) + (36 \times 2.16^2) \right] = 4.05 \times 10^{-3} \text{ m}
\]

6. We have \(x = 2.5 \sin(2\theta)\). For stationary points:
\[
\frac{dx}{d\theta} = 5 \cos(2\theta) = 0
\]
\[
\cos(2\theta) = 0, \quad 2\theta = \cos^{-1}(0) = \frac{\pi}{2} \quad \text{gives} \quad \theta = \frac{\pi}{4}
\]
Using the second derivative test:
\[
\frac{d^2x}{d\theta^2} = -10 \sin(2\theta) \quad \left[ \text{By} \quad \frac{d}{d\theta} \left[ \cos(k\theta) \right] = -k \sin(k\theta) \right]
\]
At \(\theta = \frac{\pi}{4}\), \(\frac{d^2x}{d\theta^2} = -10 \sin\left(2 \times \frac{\pi}{4}\right) = -10 < 0\).

By (7.2), when \(\theta = \pi/4\) the horizontal distance \(x\) is a maximum.

7. Similar to solution 6. We can rewrite \(x\) as:
\[
x = \frac{u^2}{2 \times 25} \left[ 2 \sin(\theta) \cos(\theta) \right] = \frac{u^2}{50} \sin(2\theta)
\]
Differentiating with respect to \(\theta\) gives:
\[
\frac{dx}{d\theta} = \frac{u^2}{50} 2 \cos(2\theta) = \frac{u^2}{25} \cos(2\theta)
\]
By solution 6 we have a stationary point at \(\theta = \pi/4\).

\[
(1.16) \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
\[
(4.53) \quad 2 \sin(x) \cos(x) = \sin(2x)
\]
\[
(7.2) \quad y' = 0, \quad y'' < 0 \quad \text{maximum}
\]
To show that $\theta = \pi/4$ gives maximum:

$$\frac{dx}{d\theta} = \frac{u^2}{25}\cos(2\theta)$$

$$\frac{d^2x}{d\theta^2} = \frac{u^2}{25}[-2\sin(2\theta)]$$

By $\frac{d}{d\theta}(\cos(k\theta)) = -k\sin(k\theta)$

At $\theta = \pi/4$, $\frac{d^2x}{d\theta^2} = -\frac{2u^2}{25}\sin\left(\frac{2\pi}{4}\right) = -\frac{2u^2}{25} < 0$ [Negative]

By (7.2), $\theta = \pi/4$ gives maximum $x$.

8. We have $s = 2 - te^{-t}$. The velocity, $v$, is found by differentiating:

$$v = \frac{ds}{dt} = 0 - \left[te^{-t}(1) - (e^{-t})(1-t)\right]$$

$$v = (t-1)e^{-t}$$

We need to differentiate $v$ with respect to $t$ to find the acceleration, $a$.

$$a = \frac{dv}{dt} = -e^{-t}(t-1) + e^{-t}(1)$$

$$= e^{-t}[-t+1+1]$$

$$= e^{-t}(2-t)$$

The graph $v = e^{-t}(t-1)$ cuts the $v$ axis at $t = 0$, therefore substituting $t = 0$

$$v = e^0(0-1) = -e^0 = -1$$

Also $v = e^{-t}(t-1)$ cuts the $t$ axis at $v = 0$,

$$e^{-t}(t-1) = 0, \ t-1 = 0 \ gives \ t = 1$$

The graph $v$ goes through $(0, -1)$ and $(1, 0)$. What happens to $v = e^{-t}(t-1)$ as $t \to \infty$? As $t \to \infty$, $v \to 0$ because $e^{-t}$ is decaying as $t$ increases.

What else can we discover about the graph?

Any stationary points and their nature.

$$v = e^{-t}(t-1)$$

$$\frac{dv}{dt} = a = (2-t)e^{-t} = 0 \ gives \ t = 2$$

There is a stationary point at $t = 2$. To identify the nature of stationary point we differentiate again

$$\frac{d^2v}{dt^2} = (-1)e^{-t} + (2-t)(-e^{-t}) = (t-3)e^{-t}$$

At $t = 2$, $\frac{d^2v}{dt^2} = (2-3)e^{-2} = -e^{-2} < 0$. By (7.2), at $t = 2$, $v$ has a maximum. The maximum value $= e^{-2}(2-1) = e^{-2}$. Also $\frac{d^2v}{dt^2} = 0$ when $t = 3$. Hence there is a general point of inflexion at $t = 3$ when $v = 2e^{-3}$. We have

(6.31) $(uv)' = u'v + uv'$
9. From chapter 5 we know the exponential function is never zero, so \( v = 4e^{-50t^2} \neq 0 \) for any values of \( t \). However as \( t \to \pm \infty \), \( v \to 0 \) because the exponential function, \( e^{-50t^2} \), decays as \( t \to \pm \infty \). We can find the stationary points: \( v = 4e^{-50t^2} \)

\[
\frac{dv}{dt} = 4e^{-50t^2}(-100t) = -400te^{-50t^2} = 0 \quad \text{gives} \quad t = 0
\]

Substituting \( t = 0 \), \( v = 4e^0 = 4 \). Hence \((0,4)\) is the stationary point of \( v = 4e^{-50t^2} \). What about the nature of the stationary point?

We can use first derivative test: \( \frac{dv}{dt} = -400te^{-50t^2} \)

If \( t < 0 \) then \( \frac{dv}{dt} > 0 \) because the exponential part \( e^{-50t^2} \) is positive and we have \(-400\) multiplied by another negative, \( t \), which gives a positive answer.

If \( t > 0 \) then \( \frac{dv}{dt} < 0 \). By (7.7) the stationary point \((0,4)\) is a maximum of \( v \).

To find general points of inflexion, we must differentiate again:

\[
\frac{d^2v}{dt^2} = -400\left( \frac{d}{dt}\left[ te^{-50t^2} \right] \right)
\]

\[
= -400\left[ e^{-50t^2} - 100t^2e^{-50t^2} \right]
\]

For inflexion, \( \frac{d^2v}{dt^2} = -400e^{-50t^2}\left[ 1-100t^2 \right] = 0 \) gives \( t^2 = \frac{1}{100} \), \( t = \pm \frac{1}{10} \). Hence

\[
v = 4e^{-50t^2}
\]

10. We have

\[
R = \frac{\ln \left( \frac{t}{t_i} \right)}{2\pi k} + \frac{1}{2\pi th} = \frac{1}{2\pi k} \left[ \ln \left( \frac{t}{t_i} \right) + \frac{t^{-1}}{h} \right] \quad \text{(Factorizing)}
\]

For stationary points we need to differentiate \( R \) with respect to \( t \):
\[
\frac{dR}{dt} = \frac{1}{2\pi} \left[ \frac{1}{k} \frac{1}{t/t_i} \left( \frac{1}{t_i} \right) - \frac{t^2}{h} \right]
\]
\[
= \frac{1}{2\pi} \left[ \frac{1}{k} \frac{1}{t} - \frac{1}{t^2 h} \right] \quad \text{(Cancelling t's)}
\]

For stationary points we need \( \frac{dR}{dt} = 0 \), thus
\[
\frac{1}{kt} - \frac{1}{t^2 h} = 0 \quad \text{(because \( \frac{1}{2\pi} \) cannot be zero)}
\]
\[
\frac{1}{kt} = \frac{1}{t^2 h} \quad \text{gives} \quad t = \frac{k}{h} \quad \text{[Transposing]}
\]

So thickness \( t = k/h \) gives a stationary point. **How do we show this value gives minimum \( R \)?** Use the second derivative test:
\[
\frac{d^2R}{dt^2} = \frac{1}{2\pi} \left[ - \frac{t^{-2}}{k} + \frac{2t^{-3}}{h} \right] = \frac{1}{2\pi} \left[ - \frac{1}{kt^2} + \frac{2}{t^3 h} \right]
\]

Substituting \( t = k/h \):
\[
\frac{d^2R}{dt^2} = \frac{1}{2\pi} \left[ \frac{-h^2}{k^3} + \frac{2h^2}{k^3} \right]
\]
\[
= \frac{1}{2\pi} \left[ \frac{h^2}{k^3} \right] > 0 \quad \text{(since \( k > 0 \))}
\]

Hence by (7.3), thickness \( t = k/h \) gives minimum resistance \( R \).

11. We have \( \alpha = \frac{n^2 + 12}{3-n} \). **How do we differentiate this?**

You can apply long division to rewrite \( \alpha \) or use the quotient rule (6.32):
\[
\frac{d\alpha}{dn} = \frac{u'v - uv'}{v^2}
\]
\[
= \frac{2n(3-n) + (n^2 + 12)}{(3-n)^2}
\]
\[
= \frac{6n - 2n^2 + n^2 + 12}{(3-n)^2}
\]
\[
= \frac{12 + 6n - n^2}{(3-n)^2}
\]

(6.32) \( (u/v)' = (u'v - uv')/v^2 \)
(7.3) \( R' = 0, \ R'' > 0 \) minimum
For \( \frac{d\alpha}{dn} = 0, \quad 12 + 6n - n^2 = 0 \) [Numerator=0]

Multiplying by \(-1\) gives the quadratic \( n^2 - 6n - 12 = 0 \)

How do we solve this?

Substituting \( a = 1, \quad b = -6 \) and \( c = -12 \) into the quadratic formula:

\[
n = \frac{6 \pm \sqrt{36 + (4 \times 12)}}{2}
\]

\[
= 7.58 \text{ or } -1.58
\]

Hence \( n = 7.58 \) (cannot have a negative gear ratio).

How can we show \( n = 7.58 \) gives maximum acceleration, \( \alpha \)?

Use the first derivative test:

\[
\frac{d\alpha}{dn} = \frac{12 + 6n - n^2}{(3-n)^2}
\]

We only need to examine the sign of the numerator because the denominator is positive.

If \( n > 7.58 \), try \( n = 8 \), then \( 12 + (6 \times 8) - 8^2 = -4 < 0 \)

If \( n < 7.58 \), try \( n = 7 \), then \( 12 + (6 \times 7) - 7^2 = 5 > 0 \)

By (7.7), \( n = 7.58 \) gives maximum acceleration.

12. Replacing \( e^x \) with the Maclaurin series expansion of (7.15) we have:

\[
e^x - 1 = \left(1 + x + x^2/2! + x^3/3! + \ldots\right) - 1
\]

\[
= x + x^2/2! + x^3/3! + \ldots
\]

\[
= x \left(1 + x/2! + x^3/3! + \ldots\right)
\]

\[
= 1 + x + x^2/2! + x^3/3! + \ldots
\]

So \( \lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots\right) = 1 \)

13. The gradient, \( m \), of the tangent is evaluated by differentiating \( y = \sin^2(x) \):

\[
\frac{dy}{dx} = 2\sin(x)\cos(x)
\]

At \( x = \frac{\pi}{4} \), \( \frac{dy}{dx} = 2\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) = 1 \). Hence \( m = 1 \). Equation of tangent is of the form \( y = x + c \). How can we find \( c \)?

At \( x = \frac{\pi}{4} \), \( y = \left[\sin\left(\frac{\pi}{4}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}\right] \), so the tangent goes through \( x = \frac{\pi}{4} \), \( y = \frac{1}{2} \).

Substituting these gives:

\[
e^x = 1 + x + x^2/2! + x^3/3! + \ldots
\]
\[ \frac{1}{2} = \frac{\pi}{4} + c \]
\[ c = \frac{1}{2} - \frac{\pi}{4} = \frac{2 - \pi}{4} = \frac{1}{4} (2 - \pi) \]

Therefore the equation of the tangent is \( y = x + \frac{1}{4} (2 - \pi) \). How do we find the equation of the normal?

The gradient of the normal = \(-1\) so the equation of the normal is of the form:
\[ y = -x + c_i \quad (** \) \]

The normal also goes through the point \( x = \frac{\pi}{4}, \ y = \frac{1}{2} \). So
\[ \frac{1}{2} = -\frac{\pi}{4} + c_i \] gives \( c_i = \frac{1}{2} + \frac{\pi}{4} = \frac{2 + \pi}{4} = \frac{1}{4} (2 + \pi) \)

Substituting \( c_i = \frac{1}{4} (2 + \pi) \) into \(**\) gives:
\[ y = -x + \frac{1}{4} (2 + \pi) = \frac{1}{4} (2 + \pi) - x \]

14. We need to differentiate \( v = kx \ln \left( \frac{1}{x} \right) \), how?

First we can rewrite \( v \) as follows:
\[ v = kx \ln \left( \frac{1}{x} \right) = kx \ln \left( x^{-1} \right) = -kx \ln \left( x \right) \]

We can differentiate \( v \) by using the product rule, (6.31):
\[ u = x, \quad w = \ln \left( x \right) \]
\[ u' = 1, \quad w' = 1/x \]

Applying (6.31)
\[ \frac{dv}{dx} = -k \left[ 1. \ln \left( x \right) + x \left( \frac{1}{x} \right) \right] = -k \left[ \ln \left( x \right) + 1 \right] \]

For stationary points this is zero, therefore
\[ -k \left[ \ln \left( x \right) + 1 \right] = 0 \]
\[ \ln \left( x \right) + 1 = 0 \quad (\text{because } k > 0) \]
\[ \ln \left( x \right) = -1 \]

How can we find \( x \) from \( \ln \left( x \right) = -1 \)?
Taking exponential of both sides gives \( x = e^{-1} \).
Differentiate again to find whether this value, \( x = e^{-1} \), gives maximum velocity.
\[ \frac{dv}{dx} = -k \left[ \ln \left( x \right) + 1 \right] \]
\[ \frac{d^2v}{dx^2} = -k \left( \frac{1}{x} \right) = -\frac{k}{x} \]

(6.31) \( (uw)' = u'w + uw' \)
Substituting \( x = e^{-1} \) gives \( \frac{d^2v}{dx^2} = -\frac{k}{e^{-1}} < 0 \) because \( k \) and \( e^{-1} \) are both positive. By (7.2) the maximum velocity occurs at \( x = e^{-1} \).

15. Substituting \( i = 5e^{-500t} \) and \( L = 2 \times 10^{-3} \) into \( v \) gives

\[
v = \left(2 \times 10^{-3}\right) \frac{d}{dt} \left(5e^{-500t}\right) = \left(2 \times 10^{-3}\right) \left(-500 \times 5e^{-500t}\right) = \left(2 \times 10^{-3}\right) \left(-2500\right)e^{-500t} = -5e^{-500t}.
\]

As \( t \to \infty, i \to 0 \) because exponential function, \( e^{-500t} \), goes to zero. We also know it is a decaying graph because of the negative sign in front of the \( 500t \). What about stationary points:

\[
i = 5e^{-500t}, \quad \frac{di}{dt} = -2500e^{-500t}
\]

Putting this to zero gives \(-2500e^{-500t} = 0\). Where is this function zero? This function cannot be zero for any real values of \( t \) because it is the exponential function so there are no stationary points.

At \( t = 0, i = 5e^0 = 5 \). Thus we have:

\[
i = 5e^{-500t} \quad v = -5e^{-500t}
\]

16. Rewriting \( F \) we have:

\[
F = \frac{Ir^2}{2} \left(x^2 + r^2\right)^{3/2}
\]

\[
\frac{dF}{dx} = \frac{Ir^2}{2} \left(-\frac{3}{2}\right) \left(x^2 + r^2\right)^{-1/2} \left(2x\right) = -\frac{3Ir^2}{2} \frac{x}{\left(x^2 + r^2\right)^{3/2}}
\]

\[
\frac{dF}{dx} = -\frac{3Ir^2}{2} \frac{x}{\left(x^2 + r^2\right)^{3/2}}
\]

Points of inflexion occurs at \( \frac{d^2F}{dx^2} = 0 \), so we need to differentiate again, how?

Use the quotient rule (6.32) with:

\[
u = x \quad v = \left(x^2 + r^2\right)^{3/2} \quad u' = 1 \quad v' = 5 \frac{\left(x^2 + r^2\right)^{3/2}}{2} 2x = 5x \left(x^2 + r^2\right)^{3/2}
\]

(6.32) \[
\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}
\]

(7.2) \[
v' = 0, \quad v'' < 0 \quad \text{maximum}
\]
Putting this to zero gives that the numerator is zero:

\[ -\frac{3Ir^2}{2} \left[ (x^2 + r^2)^{5/2} - 5x^2 \left( x^2 + r^2 \right)^{3/2} \right] = 0 \]

This can only occur if the terms inside the square brackets are zero because the current \( I \neq 0 \) and radius \( r \neq 0 \).

\[ (x^2 + r^2)^{5/2} - 5x^2 \left( x^2 + r^2 \right)^{3/2} = 0 \]

Factorizing:

\[ (x^2 + r^2)^{3/2} \left[ (x^2 + r^2) - 5x^2 \right] = 0 \]

Again only the square brackets term can be zero because \( (x^2 + r^2)^{3/2} \neq 0 \) (all terms are squared and no negative sign).

\[ (x^2 + r^2) - 5x^2 = 0 \text{ implies } r^2 - 4x^2 = 0 \text{ which gives } x = \pm \frac{r}{2} \]

Since \( x \) is distance, \( x = \frac{r}{2} \). We need to check for change of sign of \( \frac{d^2F}{dx^2} \). If \( x < \frac{r}{2} \), then \( r^2 - 4x^2 > 0 \), hence \( \frac{d^2F}{dx^2} < 0 \) because there is a negative sign outside the square brackets in \((\dagger)\).

If \( x > \frac{r}{2} \), then \( r^2 - 4x^2 < 0 \), hence \( \frac{d^2F}{dx^2} > 0 \). At \( x = \frac{r}{2} \) we have a uniform field.

17. How can we differentiate \( \eta \) with respect to \( x \)?

Use the quotient rule (6.32) with

\[ u = xs \cos \phi \quad v = L_i + xs \cos \phi + x^2L_c \]

\[ u' = s \cos \phi \quad v' = s \cos \phi + 2xL_c \]

Substituting these into (6.32) gives:

\[
\frac{d\eta}{dx} = \frac{s \cos \phi \left( L_i + xs \cos \phi + x^2L_c \right) - xs \cos \phi \left( s \cos \phi + 2xL_c \right)}{ \left( L_i + xs \cos \phi + x^2L_c \right)^2 } 
\]

\[ = \frac{L_i s \cos \phi + xs^2 \cos^2 \phi + x^2L_c s \cos \phi - xs^2 \cos^2 \phi - 2x^2sL_c \cos \phi}{\left( L_i + xs \cos \phi + x^2L_c \right)^2} \]

\[ = \frac{L_i s \cos \phi - x^2sL_c \cos \phi}{\left( L_i + xs \cos \phi + x^2L_c \right)^2} \]

\[
\frac{d\eta}{dx} = \frac{s \cos \phi \left( L_i - x^2L_c \right)}{\left( L_i + xs \cos \phi + x^2L_c \right)^2} 
\]

\( (u/v)' = (u'v - uv')/v^2 \)
For stationary point \( \frac{d\eta}{dx} = 0 \), hence the numerator = 0. Since \( s\cos(\phi) > 0 \) we have

\[
L_i - x^2 L_e = 0
\]

\[
L_i = x^2 L_e, \quad x^2 = \frac{L_i}{L_e} \text{ gives } x = \sqrt{\frac{L_i}{L_e}}
\]

*How can we show that this value of \( x \) gives maximum efficiency?*

Use the first derivative test (7.7):

\[
\frac{d\eta}{dx} = \frac{s\cos(\phi) (L_i - x^2 L_e)}{(L_i + xs\cos(\phi) + x^2 L_e)^2}
\]

We only need to examine the term \( L_i - x^2 L_e \) because the other terms are positive.

If \( x < \sqrt{\frac{L_i}{L_e}} \) then \( x^2 < \frac{L_i}{L_e} \) so \( L_i - x^2 L_e > 0 \) and \( \frac{d\eta}{dx} > 0 \)

If \( x > \sqrt{\frac{L_i}{L_e}} \) then \( x^2 > \frac{L_i}{L_e} \) so \( L_i - x^2 L_e < 0 \) and \( \frac{d\eta}{dx} < 0 \)

By (7.7), \( x = \sqrt{\frac{L_i}{L_e}} \) gives maximum efficiency.

18. (i) Let \( f(x) = \sinh(x) \) then

\[
f(x) = \sinh(x) \quad f(0) = \sinh(0) = 0 \\
f'(x) = \cosh(x) \quad f'(0) = \cosh(0) = 1 \\
f''(x) = \sinh(x) \quad f''(0) = 0 \\
f'''(x) = \cosh(x) \quad f'''(0) = 1 \\
f^{(4)}(x) = \sinh(x) \quad f^{(4)}(0) = 0 \\
f^{(5)}(x) = \cosh(x) \quad f^{(5)}(0) = 1
\]

Substituting these into (7.14) gives:

\[
\sinh(x) = 0 + (1)x + 0 + (1)\frac{x^3}{3!} + 0 + (1)\frac{x^5}{5!} + \ldots
\]

\[
= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots
\]

(ii) The MAPLE output is on the web site.
19. (i) We let \( f(x) = \tan^{-1}(x) \):
\[
 f(x) = \tan^{-1}(x) \quad f(0) = 0 \\
 f'(x) = \frac{1}{1 + x^2} \quad f'(0) = 1 \\
 f''(x) = -\frac{2x}{(1 + x^2)^2} \quad f''(0) = 0 \\
 f'''(x) = \frac{6x^2 - 2}{(1 + x^2)^3} \quad f'''(0) = -2 \\
 f^{(4)}(x) = \frac{24(x - x^3)}{(1 + x^2)^4} \quad f^{(4)}(0) = 0 \\
 f^{(5)}(x) = \frac{24(1 - 10x^2 + 5x^4)}{(1 + x^2)^5} \quad f^{(5)}(0) = 24
\]

We have 3 non-zero terms; \( f'(0) = 1 \), \( f'''(0) = -2 \) and \( f^{(5)}(0) = 24 \).
Substituting these into (7.14) gives
\[
\tan^{-1}(x) = 0 + (1 \times x) + 0 + \left(\frac{-2}{3!}\right)x^3 + 0 + \left(\frac{24}{5!}\right)x^5 + ...
\]
\[= x - \frac{x^3}{3} + \frac{x^5}{5} + ...\]
(\(*\))

(ii) To obtain the required result we need to substitute \( x = 1 \) into (\(*\)):
\[
\tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} + ... \\
\]
Remember \( \tan^{-1}(1) = \frac{\pi}{4} \). Thus
\[
\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + ...
\]

(iii) All evaluations equal \( \pi/4 \).

20. (i)
\[
y = \frac{1}{3}x^3 - 3x^2 + 8x - 3 \\
\frac{dy}{dx} = x^2 - 6x + 8
\]
For turning point \( \frac{dy}{dx} = 0 \\
x^2 - 6x + 8 = 0, \ (x - 4)(x - 2) = 0 \) gives \( x = 4 \) or \( x = 2 \)
\[
\frac{d^2y}{dx^2} = 2x - 6
\]
At \( x = 2, \ \frac{d^2y}{dx^2} = -2 < 0 \) maximum, \( y = \frac{11}{3} \)

(7.14) \[ f(x) = f(0) + f'(0)x + x^2f''(0)/2! + x^3f'''(0)/3! + ... \]
At $x = 4$, $\frac{d^2y}{dx^2} = 2 > 0$ minimum, $y = \frac{7}{3}$

The curve $y = \frac{1}{3}x^3 - 3x^2 + 8x - 3$ cuts the $y$ axis at $-3$ (the value of $y$ at $x = 0$).

(ii) Let

$$f(x) = \frac{1}{3}x^3 - 3x^2 + 8x - 3$$

$$f'(x) = x^2 - 6x + 8$$

By looking at the graph, take $r_1 = 0$ (you could just as well take $r_1 = 1$)

$$r_2 = \frac{0 - f(0)}{f''(0)} = 0.3750$$

$$r_3 = 0.375 - \frac{f(0.375)}{f''(0.375)} = 0.4437$$

$$r_4 = 0.4437 - \frac{f(0.4437)}{f''(0.4437)} = 0.4458$$

$$r_5 = 0.4458 - \frac{f(0.4458)}{f''(0.4458)} = 0.4458$$

Since $r_4 = r_5$, the root of $\frac{1}{3}x^3 - 3x^2 + 8x - 3 = 0$ is 0.446 (3 d.p.).

21. (i) We use the trapezium rule to determine the area $A$ in the given diagram

$$A = \frac{1}{2} x (4 + y) \quad (*)$$

We are given that

$y + YZ = 6$ implies that $YZ = 6 - y$
YZ can be found by Pythagoras:

\[ YZ^2 = (6 - y)^2 = (4 - y)^2 + x^2 \]

\[ 36 - 12y + y^2 = 16 - 8y + y^2 + x^2 \]

Collecting like terms gives

\[ 20 - x^2 = 4y \quad \text{which gives} \quad y = \frac{1}{4} (20 - x^2) \]

Substituting \( y = \frac{1}{4} (20 - x^2) \) into (*) yields

\[ A = \frac{1}{2} x \left( 4 + \frac{1}{4} (20 - x^2) \right) \]

\[ = \frac{1}{8} x (16 + 20 - x^2) = \frac{1}{8} x (36 - x^2) = \frac{1}{8} (36x - x^3) \]

(ii) For maximum cross-sectional area we differentiate the above function:

\[ A = \frac{1}{8} (36x - x^3) \]

\[ \frac{dA}{dx} = \frac{1}{8} (36 - 3x^2) \]

Stationary points occur where the derivative is zero:

\[ \frac{1}{8} (36 - 3x^2) = 0 \quad \Rightarrow \quad 36 - 3x^2 = 0 \quad \Rightarrow \quad x^2 = 12 \quad \Rightarrow \quad x = \sqrt{12} = 2\sqrt{3} \]

To show that we have a maximum at this value of \( x \) we differentiate again:

\[ \frac{dA}{dx} = \frac{1}{8} (36 - 3x^2) \]

\[ \frac{d^2A}{dx^2} = \frac{1}{8} (0 - 6x) \]

Substituting \( x = 2\sqrt{3} \) into \( \frac{d^2A}{dx^2} = \frac{1}{8} (0 - 6x) = -\frac{6}{8} x \) gives a negative value so we have maximum at \( x = 2\sqrt{3} \). We can substitute this value into \( y = \frac{1}{4} (20 - x^2) \) to find \( y \):

\[ y = \frac{1}{4} (20 - x^2) = \frac{1}{4} (20 - (2\sqrt{3})^2) = 2 \]

Hence \( x = 2\sqrt{3} \) m and \( y = 2 \) m gives maximum cross-sectional area.

22. Using the binomial series, (7.24), with \( x = -\frac{v}{c^2} \) we have

\[
\left( 1 - \frac{v^2}{c^2} \right)^{\frac{1}{2}} = 1 + \frac{1}{2} \left( -\frac{v^2}{c^2} \right) + \left[ \frac{1}{2} \left( -\frac{v^2}{c^2} \right) \right] \left( -\frac{v^2}{c^2} \right)^2 + \left[ \frac{1}{2} \left( -\frac{v^2}{c^2} \right) \right] \left( -\frac{v^2}{c^2} \right)^3 + ... 
\]

\[ (7.24) \]

\[ (1 + x)^n = 1 + nx + \left[ \frac{n(n-1)}{2!} \right] x^2 + \left[ \frac{n(n-1)(n-3)}{3!} \right] x^3 + ... 
\]

\[ (7.29) \]

\[ r_{n+1} = r_n + \frac{f(r_n)}{f'(r_n)} \]
23. Similar to solution of question 22 but we ignore higher powers.
By using the binomial expansion we can show that
\[ \frac{1}{\sqrt{1-x}} = (1-x)^{-1/2} = 1 + \frac{1}{2} x + \frac{3}{8} x^2 + \frac{5}{16} x^3 + \ldots \]

Substituting \( x = \left( \frac{v}{c} \right)^2 \) because we are given \( m = \frac{m_0}{\sqrt{1-\left( \frac{v}{c} \right)^2}} \) into the above:

\[
m = \frac{m_0}{\sqrt{1-\left( \frac{v}{c} \right)^2}} = \frac{m_0}{\sqrt{1-\left( \frac{v}{c} \right)^2}} = m_0 \left( 1 + \frac{1}{2} \left( \frac{v}{c} \right)^2 + \frac{3}{8} \left( \frac{v}{c} \right)^4 + \frac{5}{16} \left( \frac{v}{c} \right)^6 + \ldots \right)
\]

We are told that \( v \) is very small compared to \( c \) therefore \( \left( \frac{v}{c} \right) \) is a small number and taking powers makes it even smaller. Hence we ignore the higher powers of \( \left( \frac{v}{c} \right) \), that is powers above 2. Hence we have

\[
m = m_0 \left( 1 + \frac{1}{2} \left( \frac{v}{c} \right)^2 \right) = m_0 \left( 1 + \frac{1}{2} \frac{v^2}{c^2} \right)
\]

Substituting this into the given formula for KE, \( K = (m-m_0)c^2 \), we have

\[
K = \left( m_0 \left( 1 + \frac{1}{2} \frac{v^2}{c^2} \right) - m_0 \right) c^2
\]

\[
= \left( m_0 + \frac{1}{2} \frac{v^2}{c^2} m_0 - m_0 \right) c^2 = \left( \frac{1}{2} \frac{v^2}{c^2} m_0 \right) c^2 = \frac{1}{2} m_0 v^2
\]

This is our required result.