Complete solutions to Exercise 3(d)

1. (i) \( f(2x + 3) = (2x + 3) + 1 = 2x + 4 \)
   (ii) \( f(x) \cdot g(x) = (x + 1)(2x + 3) = 2x^2 + 3x + 2x + 3 \)
       \( = 2x^2 + 5x + 3 \)
   (iii) \( g - 1 = (2x + 3) - 1 = 2x + 2 \)
   (iv) \( f + g = (x + 1) + (2x + 3) = 3x + 4 \)
   (v) \( f - g = (x + 1) - (2x + 3) = x + 1 - 2x - 3 = -x - 2 \)
   (vi) \( \frac{f}{g} = \frac{x + 1}{2x + 3} \) \( (2x + 3 \neq 0) \)

2. (i) \( f \circ f = f(f(x)) = f(2x + 3) = 2(2x + 3) + 3 \)
   \( = 4x + 9 \)
   (ii) \( f(f(f(x))) = f(4x + 9) = 2(4x + 9) + 3 = 8x + 21 \)
   because \( f(f(x)) = 4x + 9 \) by part (i)
   (iii) \( f(f(-3)) \approx [8 \times (-3)] + 21 = -3 \)

3. (i) \( f(0) = (a \times 0^2) + (b \times 0) + c = c \)
   (ii) Replace \( x \) with \( x + 1 \) into \( f(x) \):
       \( f(x + 1) = a(x + 1)^2 + b(x + 1) + c \)
       \( = a(x^2 + 2x + 1) + bx + b + c \)
       \( = ax^2 + 2ax + a + bx + b + c \)
       \( = ax^2 + (2a + b)x + a + b + c \)
   (iii) \( f(x + 1) - f(x) = ax^2 + 2ax + bx + a + b + c - (ax^2 + bx + c) \)
       \( = ax^2 + 2ax + bx + a + b + c - ax^2 - bx - c \)
       \( = 2ax + a + b \)

4. (i) \( g(g(x)) = g(x + 1) = (x + 1) + 1 = x + 2 \)
   (ii) \( f(g(x)) = f(x + 1) = (x + 1)^2 \)
       \( \approx x^2 + 2x + 1 \)
   (iii) \( g(f(x)) = g(x^2) = x^2 + 1 \)

5. (i) \( f(g(x)) = f(x) = x^2 - 1 \)
   (ii) \( g(f(x)) = g(x^2 - 1) = x^2 - 1 = f(x) \)
   (iii) Since \( g \) is the identity function \( f \circ g = g \circ f = f \).
   Note: It is always the case if \( f \) or \( g \) are identity functions then \( f \circ g = g \circ f \).
   (iv) \( g \circ g = g(x) = x \)

\( (a + b)^2 = a^2 + 2ab + b^2 \)
(v) We have
\[ f \circ f = f(x^2 - 1) = (x^2 - 1)^2 - 1 \]
\[ = \frac{x^4 - 2x^2 + 1}{x^2 - 1} \]
by (1.14)
\[ = x^4 - 2x^2 = x^2(x^2 - 2) \]

6. (i) Let \( y = f(x) \) and then transpose to make \( x \) the subject:

\[ y = \frac{6}{3 - x} \]
\[ 3y - xy = 6 \]
\[ 3y - 6 = xy \]
\[ x = \frac{3y - 6}{y} \]
\[ f^{-1}(x) = \frac{3x - 6}{x} \quad (x \neq 0) \]

(ii)
\[ f \circ f^{-1} = f\left(\frac{3x - 6}{x}\right) = \frac{6}{3 - \left(\frac{3x - 6}{x}\right)} \quad \text{multiply numerator and denominator by } x \]
\[ = \frac{6x}{3x - 3x + 6} = \frac{6x}{6} = x \]

(iii)
\[ f^{-1} \circ f = f^{-1}\left(\frac{6}{3 - x}\right) = \frac{3\left(\frac{6}{3 - x}\right) - 6}{6f(3 - x)} \]
\[ = \frac{18 - 6(3 - x)}{6} \quad \text{(multiply numerator and denominator by } 3 - x) \]
\[ = \frac{18 - 18 + 6x}{6} = \frac{6x}{6} = x \]

7. (i) \( f(f^{-1}(x)) = f\left(\frac{2x - 3}{1 + x}\right)^3 \)
\[ = \frac{2x - 3 + 3}{2 - 2x - 3} \]
\[ = \frac{2x - 3 + 3(1 + x)}{2(1 + x) - (2x - 3)} \]
\[ = \frac{2x - 3 + 3x}{2 + 2x - 2x + 3} \]
\[ = \frac{5x}{5} = x \]

(1.14) \( (a - b)^2 = a^2 - 2ab + b^2 \)
\[
(f^{-1} \circ f)(x) = f^{-1}\left(\frac{x^3 + 3}{2 - x^3}\right)
\]

\[
= f^{-1}\left(\frac{2\left(\frac{x^3 + 3}{2 - x^3}\right) + 3}{2 - x^3}\right)
\]

\[
= \frac{2x^3 + 6 - 3(2 - x^3)}{2 - x^3 + x^3 + 3}
\]

\[
= \frac{2x^3 + 6 - 3}{5x^3}
\]

\[
= \frac{5x^3}{5}
\]

8. We have \(f \circ f^{-1}(x) = f^{-1} \circ f(x) = x\) (identity function). This is generally the case.

9. (i) \(\sqrt{x} - 1 = f\left(\sqrt{x}\right) = f\left(g(x)\right) = f \circ g\) or \(\sqrt{x} - 1 = g(x) - 1\). Hence \(f \circ g = g - 1\)

(ii) We have \(\sqrt{x} - 1 = g(x - 1) = g(f(x)) = g \circ f\)

(iii) We have \(\sqrt{x} - 1 + 7 = g \circ f + 7\)

(iv) \(\sqrt{x^2 - 2x + 1} = \sqrt{(x - 1)^2} = x - 1 = f\) or \(\sqrt{(x - 1)^2} = g((x - 1)^2) = g(f^2)\)

10. Very similar to EXAMPLE 16; \(F(t) = \frac{t}{5}, R(t) = 1 - \frac{t}{5}\) and \(h(t) = \frac{1}{5 - t}\).

11. (a) Substituting \(G(s) = \frac{k}{s(s + 1)}\) and \(N(s) = 0.01\) into

\[
T(s) = \frac{G(s)}{1 + N(s)G(s)}
\]

gives

\[
T(s) = \frac{k}{s + 1} \left(\frac{k}{s + 1}\right)
\]

Simplify (†) by multiplying numerator and denominator by \(s(s + 1)\):

\[
T(s) = \frac{k}{s(s + 1) + 0.01k}
\]

(b) Similarly for (b) we multiply numerator and denominator by \(s + k_1\)
(c) Note that we can factorize the denominator of $G(s)$:

$$G(s) = \frac{s + 1}{s^2 + 3s + 2} = \frac{s + 1}{(s + 2)(s + 1)} = \frac{1}{s + 2} \quad \text{(cancelling } s + 1)$$

We have $G(s) = \frac{1}{s + 2}$ and $N(s) = 0.3$, this is of the form of (b) where $G(s) = \frac{1}{s + k_1}$ and $N(s) = k_2$. Here we have $k_1 = 2$ and $k_2 = 0.3$. Substituting $k_1 = 2$ and $k_2 = 0.3$ into the result for (b),

$$T(s) = \frac{1}{s + (k_1 + k_2)} = \frac{1}{s + 2.3}$$

12.

$$G(s) = \frac{10s}{(s - 2)(s^2 + 2s - 5)} = \frac{10s}{s^3 + 2s^2 - 5s - 2s^2 - 4s + 10} = \frac{10s}{s^3 - 9s + 10}$$

$$\frac{G(s)}{1 + G(s)H(s)} = \frac{10s(s^3 - 9s + 10)}{1 + \left(\frac{10s}{s^3 - 9s + 10}\right) \times (s + 3)} = \frac{10s(s^3 - 9s + 10)}{s^3 - 9s + 10 + 10s(s + 3)} \quad \text{(multiplying top and bottom by } s^3 - 9s + 10)$$

$$= \frac{10s}{s^3 - 9s + 10 + 10s^2 + 30s} = \frac{10s}{s^3 + 10s^2 + 21s + 10}$$