Solutions 7(b) 1

Complete solutions to Exercise 7(b)

1. Since the perimeter = 100 we have

\[2x + 2y = 100\]

\[x + y = 50\]  \quad [\text{Dividing both sides by 2}]

\[y = 50 - x\]  \quad (*)

The area \(A = xy\), substituting \(y = 50 - x\) gives:

\[A = x(50 - x)\]

\[A = 50x - x^2\]

Differentiating to find the stationary point:

\[\frac{dA}{dx} = 50 - 2x = 0 \quad \text{gives} \quad x = 25m\]

\[\frac{d^2A}{dx^2} = -2 < 0\]

By (7.2), \(x = 25m\) gives maximum area. Substituting this value into (*):

\[y = 50 - 25 = 25m\]

It seems as if the area constraint by the fence needs to be a square, \(x = y = 25m\).

2. From the fencing of 240m around the field we have

\[2x + y = 240\]

\[y = 240 - 2x\]  \quad (†)

The area \(A\) is given by

\[A = xy\]

\[= x(240 - 2x) \quad \text{by (†)}\]

\[= 240x - 2x^2\]

For stationary points:

\[\frac{dA}{dx} = 240 - 4x = 0, \quad 4x = 240, \quad x = 60m\]

To show maximum we need to differentiate again

\[\frac{d^2A}{dx^2} = -4 < 0\]

By (7.2), when \(x = 60m\) we have maximum area. To find \(y\) we substitute \(x = 60\) into (†):

\[y = 240 - (2 \times 60) = 120m\]

Observe that \(y\) is twice the length of \(x\), \(x = 60m\) and \(y = 120m\) gives maximum area.

3. We can rewrite \(\frac{C_D + kC_L^2}{C_L}\) as

\[\frac{C_D}{C_L} + k\frac{C_L^2}{C_L} = C_D C_L^{-1} + kC_L\]

Differentiating this with respect to \(C_L\) gives:

\[\frac{d}{dC_L} \left(C_D C_L^{-1} + kC_L\right) = -C_D C_L^{-2} + k = k - C_D C_L^{-2}\]

\[(7.2) \quad A' = 0, \quad A'' < 0 \quad \text{maximum}\]

For stationary point this is equal to zero:
\[ k - C_D C_L^2 = 0 \quad \text{gives} \quad k = C_D C_L^2 = \frac{C_D}{C_L^2} \]

Rearranging gives \( C_L^2 = \frac{C_D}{k} \). Taking the square root of both sides:

\[ C_L = \sqrt{\frac{C_D}{k}} = \left( \frac{C_D}{k} \right)^{\frac{1}{2}} \]

To check that this value of \( C_L \) gives a minimum we need to differentiate again:

\[ \frac{d}{dC_L} \left( k - C_D C_L^{-2} \right) = -2C_D C_L^{-3} = \frac{2C_D C_L^3}{C_L^3} = 2 \left( \frac{k^{3/2}}{C_D} \right)^{3/2} \]

Substituting \( C_L = \left( \frac{C_D}{k} \right)^{1/2} \) into \((\dagger)\) gives:

\[ 2C_D \left[ \left( \frac{C_D}{k} \right)^{1/2} \right]^{3/2} = \frac{2C_D}{\left( \frac{C_D}{k} \right)^{3/2}} = \frac{2}{C_D^{1/2} / k^{3/2}} = \frac{2k^{3/2}}{C_D^{1/2}} = \frac{2 \left( k^{3} \right)^{1/2}}{C_D^{1/2}} = 2 \left( \frac{k^{3}}{C_D} \right)^{1/2} \]

\[ 2 \left( \frac{k^{3}}{C_D} \right)^{1/2} = 2 \sqrt{\frac{k^{3}}{C_D}} > 0 \quad \text{(taking the positive square root)} \]

Hence by (7.3) when \( C_L = \sqrt{\frac{C_D}{k}} \) we have minimum drag.

4. We have

\[ \pi r^2 h = 8 \]
\[ h = \frac{8}{\pi r} \]

(\dagger)

The surface area, \( A \), is given by the base, \( \pi r^2 \), and the curved area, \( 2\pi rh \), hence

\[ A = \pi r^2 + 2\pi rh = \pi r^2 + 2\pi r \left( \frac{8}{\pi r} \right) = \pi r^2 + \frac{16}{r} \]

\[ A = \pi r^2 + 16r^{-1} \]

For stationary points:

\[ \frac{dA}{dr} = 2\pi r - 16r^{-2} = 0 \]

\[ 2\pi r = \frac{16}{r^2} \]

\[ r^3 = \frac{16}{2\pi} = \frac{8}{\pi} \]

Take the cube root of both sides: \( r = \sqrt[3]{\frac{8}{\pi}} = \frac{2}{\pi^{1/3}} \)

To find whether this stationary point is a maximum or minimum we use the second derivative test:

\[ \frac{dA}{dr} = 2\pi r - 16r^{-2} \quad \frac{d^2A}{dr^2} = 2\pi + 32r^{-3} \]

(7.3) \quad A’ = 0, \ A” > 0 \quad \text{minimum}
Substituting \( r = \frac{2}{\pi^{\frac{1}{3}}} \) into \( \frac{d^2A}{dr^2} \) gives \( \frac{d^2A}{dr^2} > 0 \). By (7.3), when \( r = \frac{2}{\pi^{\frac{1}{3}}} m \) we have a minimum surface area. How do we find \( h \)?

Substitute \( r = \frac{2}{\pi^{\frac{1}{3}}} \) into \( \dagger \):

\[
h = \frac{8}{\pi \left( 2/\pi^{\frac{1}{3}} \right)^2} = \frac{8}{\pi \left( 2^2/\pi^{\frac{2}{3}} \right)} = \frac{2}{\pi^{\frac{2}{3}}}
\]

The height and radius are equal.

5. Similar to **Example 9** with 1000 replaced by \( V \). Volume \( V = \pi r^2 h \), gives

\[
h = \frac{V}{\pi r^2} \quad (\ast)
\]

**Surface Area** \( A = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r \left( \frac{V}{\pi r^2} \right) \)

\[
A = 2\pi r^2 + 2V r^{-1}
\]

\[
\frac{dA}{dr} = 4\pi r - 2V r^{-2} = 0
\]

\[
4\pi r = \frac{2V}{r^2}
\]

\[
r^3 = \frac{2V}{4\pi} = \frac{V}{2\pi}, \quad r = \sqrt[3]{\frac{V}{2\pi}} = \left( \frac{V}{2\pi} \right)^{\frac{1}{3}}
\]

To show we have minimum surface area we have to differentiate again

\[
\frac{d^2A}{dr^2} = 4\pi + 4V r^{-3}
\]

\[
= 4\pi + \frac{4V}{r^3} > 0
\]

\( \frac{d^2A}{dr^2} \) is going to be positive because \( r \) is radius and is therefore positive.

By (7.3), when \( r = \left( \frac{V}{2\pi} \right)^{\frac{1}{3}} \) we have minimum surface area.

To find \( h \) we substitute \( r = \left( \frac{V}{2\pi} \right)^{\frac{1}{3}} \) into \( (\ast) \) and obtain \( h = 2r \).

6. Factorizing the given equation gives:

\[
M = \frac{W}{2} \left( Lx - x^2 \right)
\]

\[
\frac{dM}{dx} = \frac{W}{2} \left( L - 2x \right) = 0, \quad \text{so } L - 2x = 0 \text{ which gives } x = \frac{L}{2}
\]

\[
\frac{d^2M}{dx^2} = \frac{W}{2} (-2) = -W < 0
\]

(7.2) \( M' = 0, \ M'' < 0 \) maximum

(7.3) \( A' = 0, \ A'' < 0 \) minimum
By (7.2), the bending moment is a maximum at \( x = \frac{L}{2} \). The maximum value of \( M \) is evaluated by substituting \( x = \frac{L}{2} \) into \( M \):

\[
M = \frac{W}{2} \left( Lx - x^2 \right)
\]

\[
= \frac{W}{2} \left( \frac{L}{2} - \frac{L}{2} \right)
\]

\[
= \frac{W}{2} \left[ \frac{L^2}{2} - \frac{L^2}{4} \right] = \frac{W}{2} \left( \frac{L^2}{4} \right) = \frac{WL^2}{8}
\]

7. We have

\[
y = \frac{W}{6EI} \left( 3L^3x - x^3 \right) \tag{*}
\]

For stationary points

\[
\frac{dy}{dx} = \frac{W}{6EI} \left( 3L^2 - 3x^2 \right) = 0, \quad 3L^2 - 3x^2 = 0, \quad x^2 = L^2, \quad x = \pm L
\]

\( x \) cannot be \(-L\) because \( L \) is a length. Hence \( x = L \).

For maximum;

\[
\frac{d^2y}{dx^2} = \frac{W}{6EI} \left( -6x \right)
\]

\( x = L \)

\[
\frac{d^2y}{dx^2} = -\frac{W}{EI} L < 0
\]

By (7.2), at \( x = L \) we have maximum deflection.

Maximum deflection can be evaluated by substituting \( x = L \) into (*)

\[
y = \frac{W}{6EI} \left( 3L^3L - L^3 \right)
\]

\[
= \frac{W}{6EI} \left( 3L^3 - L^3 \right)
\]

\[
= \frac{W}{6EI} \left( 2L^3 \right) = \frac{WL^3}{3EI}
\]

8. Clearly by looking at Fig 20 it looks as if the maximum deflection will occur furthest from the fixed end, \( x = L \). We need to show this by using differentiation.

\[
y = \frac{W}{24EI} \left[ x^4 - 4Lx^3 + 4L^2x^2 \right]
\]

\[
\frac{dy}{dx} = \frac{W}{24EI} \left[ 4x^3 - 12Lx^2 + 8L^2x \right] \equiv \frac{4Wx}{24EI} \left[ x^2 - 3Lx + 2L^2 \right] = 0
\]

\( x = 0 \) or \( x^2 - 3Lx + 2L^2 = 0 \)

\( (x - 2L)(x - L) = 0 \)

\( x = 2L, \ x = L \)

We have a stationary points at \( x = 0, x = L \) and \( x = 2L \).

\( x \) cannot be \( 2L \) because the cantilever is only of length \( L \).

Also at \( x = 0 \), the fixed end, there is no deflection (minimum). We need to show that at \( x = L \) we do have maximum deflection.

(7.2) \( y' = 0, \ y'' < 0 \) maximum
\begin{align*}
\frac{dy}{dx} &= \frac{W}{24EI} \left[ 4x^3 - 12Lx^2 + 8L^2 x \right] \\
\frac{d^2 y}{dx^2} &= \frac{W}{24EI} \left[ 12x^2 - 24Lx + 8L^2 \right]
\end{align*}

Substituting \( x = L \) gives \( \frac{d^2 y}{dx^2} = \frac{W}{24EI} \left[ 12L^2 - 24L^2 + 8L^2 \right] = \frac{W}{24EI} [-4L^2] < 0 \)

By (7.2), at \( x = L \) we have maximum deflection. The value of this deflection is \( y = \frac{W}{24EI} \left[ L^4 - 4L^4 + 4L^2 L^2 \right] = \frac{WL^4}{24EI} \)

9. We have
\[
p = T v - mv^3
\]
\[
\frac{dp}{dv} = T - 3mv^2 = 0, \quad 3m v^2 = T \quad \text{transposing gives} \quad v = \sqrt{\frac{T}{3m}}
\]

To show maximum for this value of \( v \):
\[
\frac{d^2 p}{dv^2} = -6mv = -6m \sqrt{\frac{T}{3m}} < 0
\]

Hence by (7.2), \( v = \sqrt{\frac{T}{3m}} \) gives maximum power.

10. (i) \( T = 3\sin(2\theta) + 6\sin(\theta) \)
\[
\frac{dT}{d\theta} = 6\cos(2\theta) + 6\cos(\theta)
\]
\[
= 6 \left( 2\cos^2(\theta) - 1 \right) + 6\cos(\theta)
\]
\[
= 12\cos^2(\theta) + 6\cos(\theta) - 6
\]

We need to solve \( 12\cos^2(\theta) + 6\cos(\theta) - 6 = 0 \)

Let \( x = \cos(\theta) \):
\[
12x^2 + 6x - 6 = 0 \quad 2x^2 + x - 1 = 0 \quad \text{[Dividing by 6]}
\]

\[
(2x - 1)(x + 1) = 0 \quad \text{gives} \quad x = 1/2 \quad \text{or} \quad x = -1
\]

If \( \cos(\theta) = 1/2 \) then \( \theta = \pi/3 \). If \( \cos(\theta) = -1 \) then \( \theta \) lies outside the range \( 0 \leq \theta < \pi \).

So \( \theta \) can only be \( \pi/3 \).

\[
\frac{d^2 T}{d\theta^2} = -12\sin(2\theta) - 6\sin(\theta)
\]

At \( \theta = \pi/3 \), \( \frac{d^2 T}{d\theta^2} = -12\sin\left( \frac{2\pi}{3} \right) - 6\sin\left( \frac{\pi}{3} \right) < 0 \)

By (7.2), at \( \theta = \pi/3 \) we have maximum torque.

(4.54) \quad \cos(2x) = 2\cos^2(x) - 1
(7.2) \quad y' = 0, \quad y'' < 0 \quad \text{maximum}
(ii) \( T = 3 \sin \left( \frac{2\pi}{3} \right) + 6 \sin \left( \frac{\pi}{3} \right) = \frac{9\sqrt{3}}{2} \approx 7.8 \text{Nm} \)

11. (i) Similar to \textbf{EXAMPLE 10}.
\[
P = i_1^2 R_1 + (i - i_1)^2 R_2
\]
For stationary points:
\[
\frac{dP}{di_1} = 2i_1 R_1 + 2 (i - i_1) R_2 (-1) = 0
\]
\[
i_1 = \frac{iR_2}{R_1 + R_2} \quad \text{(*)}
\]
To establish this stationary point is a minimum, use the 2nd derivative:
\[
\frac{d^2P}{di_1^2} = 2R_1 + 2R_2 > 0 \quad \text{(because } R_1 > 0 \text{ and } R_2 > 0 \text{)}
\]
By (7.3) (*) gives \( P \) min.
(ii) Substitute for \( i_1 \) from (*) into \( iR = i_1 R_1 \) gives the required result.