Complete solutions to Exercise 8(c)

1. (a) \[ \int \sin(7x+1)\,dx = -\frac{\cos(7x+1)}{7} + C \quad \text{(by (8.39))} \]

(b) \[ \int \cos(7x+1)\,dx = \frac{\sin(7x+1)}{7} + C \quad \text{(by (8.39))} \]

2. Using (8.38):
   (a) \[ \int \cos(\omega t)\,dt = \frac{\sin(\omega t)}{\omega} + C \]
   (b) \[ \int \cos(\omega t + \theta)\,dt = \frac{\sin(\omega t + \theta)}{\omega} + C \]

3. (a) \[ \int \sin(\omega t)\,dt = \frac{-\cos(\omega t)}{\omega} + C \quad \text{(by (8.39))} \]
   (b) \[ \int \sin(\omega t)\,d(\omega t) = -\cos(\omega t) + C \quad \text{(by (8.7) with } u = \omega t) \]

4. (a) Differentiating \( x^2 - 1 \) with respect to \( x \) gives \( 2x \). Hence by using (8.42) we have

\[ \int \frac{-2x}{x^2 - 1}\,dx = \ln|x^2 - 1| + C \]

(b) Differentiating \( x^3 - 3x^2 + 1 \) with respect to \( x \) gives \( 3x^2 - 6x \). Using (8.42)

\[ \int \frac{3x^2 - 6x}{x^3 - 3x^2 + 1}\,dx = \ln|x^3 - 3x^2 + 1| + C \]

5. We have \( \cot(x) = \frac{\cos(x)}{\sin(x)} \). Differentiating \( \sin(x) \) gives \( \cos(x) \), hence

\[ \int \cot(x)\,dx = \int \frac{\cos(x)}{\sin(x)}\,dx \]

\[ \equiv \ln|\sin(x)| + C \quad \text{(by (8.42))} \]

6. (a) We have \( \tanh(x) = \frac{\sinh(x)}{\cosh(x)} \). Differentiating \( \cosh(x) \) gives \( \sinh(x) \), so we can use (8.42):

\[ \int \tanh(x)\,dx = \int \frac{\sinh(x)}{\cosh(x)}\,dx \equiv \ln|\cosh(x)| + C \]

(b) We have \( \coth(x) = \frac{\cosh(x)}{\sinh(x)} \) and differentiating \( \sinh(x) \) gives \( \cosh(x) \). So

\[ \int \coth(x)\,dx = \int \frac{\cosh(x)}{\sinh(x)}\,dx \equiv \ln|\sinh(x)| + C \quad \text{(by (8.42))} \]

(8.7) \[ \int \sin(u)\,du = -\cos(u) \]

(8.38) \[ \int \cos(kx + m)\,dx = \sin(kx + m)/k \]

(8.39) \[ \int \sin(kx + m)\,dx = -\cos(kx + m)/k \]

(8.42) \[ \int f'(x)/f(x)\,dx = \ln|f(x)| \]

7. (a) Differentiating \( 7t - 1 \) gives \( 7 \), so
\[
\int \frac{dt}{7t-1} = \frac{1}{7} \int \frac{7dt}{7t-1} \\
= \frac{1}{7} \ln |7t-1| + C
\] (by (8.42))

(b) Differentiating \(t^4 - 1\) with respect to \(t\) gives \(4t^3\). We can write \(t^3\) on the numerator as \(\frac{1}{4}(4t^3)\). Hence

\[
\int \frac{t^3}{t^4 - 1} dt = \frac{1}{4} \int \left( \frac{4t^3}{t^4 - 1} \right) dt \\
= \frac{1}{4} \ln |t^4 - 1| + C
\] (by (8.42))

(c) Differentiating \(5 - t^3\) with respect to \(t\) gives \(-3t^2\). We can write \(t^2\) as \(-\frac{1}{3}(-3t^2)\). Thus

\[
\int \frac{t^2}{5 - t^3} dt = -\frac{1}{3} \int \frac{-3t^2}{5 - t^3} dt \\
= -\frac{1}{3} \ln |5 - t^3| + C
\] (by (8.42))

8. We use \(\int e^{kx+m} dx = e^{kx+m}/k + C\) in each case.

(a) \(\int e^{11x+5} dx = \frac{e^{11x+5}}{11} + C\)

(b) \(\int e^{-2x+1000} dx = \frac{e^{-2x+1000}}{-2} + C = -\frac{e^{-2x+1000}}{2} + C\)

9. We have

\[v = \int (-g) dt = -gt + C\] (†)

Substituting \(t = 0,\ v = v_0\) gives

\[v_0 = -(g \times 0) + C = 0 + C,\ v_0 = C\]

Substituting \(C = v_0\) into (†):

\[v = v_0 - gt\]

10. Same as solution 9 with \(v_0 = u\).

11. Taking out the 10 gives:

\[s = 10 \int (30t + 1)^{-\frac{1}{2}} dt\] (*)&

How do we integrate \((30t + 1)^{-\frac{1}{2}}\)?

Use substitution. Let \(u = 30t + 1\), remember we also need to replace the \(dt\), how?

Differentiating:

\[u = 30t + 1,\ \frac{du}{dt} = 30\ gives\ \ dt = \frac{du}{30}\]

Putting \(u = 30t + 1\) and \(dt = \frac{du}{30}\) into (*) gives:

\[
\int \frac{f'(x)}{f(x)} dx = \ln |f(x)|
\] (8.42)
\[ s = 10 \int u^{-\frac{1}{2}} \frac{du}{30} \]
\[ = \frac{10}{30} \int u^{-\frac{1}{2}} du \]
\[ = \frac{1}{3} \left( \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right) + C \]
\[ = \frac{1}{3} \left( \frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right) + C \]
\[ = \frac{1}{3} (2u) + C \]
\[ = \frac{2}{3} u^{\frac{1}{2}} + C \]
\[ s = \frac{2}{3} (30t + 1)^{\frac{1}{2}} + C \quad \text{(Replacing } u) \]

Using \( t = 0, \ s = 2/3 \) gives:
\[ \frac{2}{3} = \frac{2}{3} \left[ (30 \times 0) + 1 \right]^{\frac{1}{2}} + C \]
\[ \frac{2}{3} = \frac{2}{3} + C \quad \text{gives } C = 0 \]

Hence \( s = \frac{2}{3} (30t + 1)^{\frac{1}{2}}. \)

12. (i)
\[ v = -6 \int t \, dt = -6 \left( \frac{t^2}{2} \right) + C \]
\[ v = -3t^2 + C \]

Substituting \( t = 0, \ v = 48 \)
\[ 48 = 0 + C \quad \text{gives } C = 48 \]

Hence \( v = 48 - 3t^2. \)

(ii) We need to find \( t \) for \( v = 0. \)
\[ 48 - 3t^2 = 0, \ 3t^2 = 48 \] which gives \( t = \sqrt{16} = 4 \text{ sec} \)

13. Rearranging \( k = \frac{P}{\rho^\gamma} \) we have \( \rho^\gamma = \frac{P}{k}. \) How can we find \( \rho \) on its own?

Taking \( \gamma \) root of both sides:
\[ \rho = \left( \frac{P}{k} \right)^{\frac{1}{\gamma}} = \frac{P^{\frac{1}{\gamma}}}{k^{\frac{1}{\gamma}}} = \frac{P^{\frac{1}{\gamma}}}{C} \quad \text{where } C = k^{\frac{1}{\gamma}} \]

Warning: This \( C \) is not the constant of integration. Putting \( \rho = \frac{P^{\frac{1}{\gamma}}}{C} \) gives:

\[ \int u^n du = \frac{u^{n+1}}{n+1} \quad \text{(8.1)} \]
\[
\int \frac{dP}{\rho} = \int \frac{dP}{P^{\gamma/\rho}} \\
= \int \frac{CdP}{P^{\gamma/\rho}} \\
= C \int P^{-\gamma/\rho} dP \\
= C \left( \frac{P^{1-\gamma/\rho+1}}{-1/\gamma + 1} \right) + D \quad (\dagger)
\]

We use \(D\) as the constant of integration. Of course we can use any letter to represent the constant of integration. Simplifying \(-1/\gamma + 1:\)

\[
\frac{1}{\gamma} + 1 = 1 - \frac{1}{\gamma} \\
= \frac{\gamma - 1}{\gamma}
\]

Replacing \(-1/\gamma + 1\) with \(\gamma - 1/\gamma\) in (\dagger) gives:

\[
\int \frac{dP}{\rho} = C \left( \frac{P^{\gamma-1}}{\gamma^{\gamma-1}} \right) + D \\
= \frac{C\gamma P^{\gamma - 1}}{\gamma - 1} + D \\
= k^{1/\gamma} \frac{P^{\gamma - 1}}{\gamma - 1} + D
\]

\[\text{(8.1) } \int u'' du = \frac{u^{n+1}}{n+1}\]