Complete solutions to Exercise 13(d)

1. Very similar to EXAMPLE 13.

(i) We have $A_0 = \pi \times 0.1^2$. Cross-section area of tank $A = 2^2 = 4$. Applying (13.10)

\[
\frac{dh}{dt} = -\pi \times 0.1^2 \frac{\sqrt{2gh}}{4} = -\left(34.79 \times 10^{-3}\right)h^{1/2}
\]

(ii) Separating variables

\[
\frac{dh}{h^{1/2}} = -\left(34.79 \times 10^{-3}\right)dt
\]

Integrating gives

\[
2h^{1/2} = -\left(34.79 \times 10^{-3}\right)t + C
\]

Substituting $t = 0$, $h = 1.3$ gives $2 \times 1.3^{1/2} = C$. Thus

\[
h^{1/2} = \frac{-\left(34.79 \times 10^{-3}\right)t + 2(1.3)^{1/2}}{2} = -\left(17.395 \times 10^{-3}\right)t + 1.14
\]

Squaring both sides

\[
h = \left[-\left(17.395 \times 10^{-3}\right)t + 1.14\right]^2
\]

(iii) Need to find $t$ for $h = 0$.

\[
0 = -\left(17.395 \times 10^{-3}\right)t + 1.14
\]

\[
t = \frac{1.14}{17.395 \times 10^{-3}} = 65.5
\]

In minutes, $t = \frac{65.5}{60} = 1.09$ mins.

(iv) The MAPLE commands are given in section D. The graph has the same shape as Fig 21 and note the height, $h$, decreases with time, $t$, and $h = 0$ when $t \geq 65.5$. 

\[
\frac{dh}{dt} = -\frac{A_0}{A \sqrt{2gh}}
\]
2. (i) The area of the square outlet is $A_0 = 0.05^2$. Cross-sectional area of tank is $A = 0.75^2 \pi$. By (13.10)

\[
\frac{dh}{dt} = -\frac{0.05^2}{0.75^2 \pi} \sqrt{2 \times 9.81 \times h} = -\left(6.266 \times 10^{-3}\right) h^{1/2}
\]

(ii) Separating variables

\[
\frac{dh}{h^{1/2}} = -\left(6.266 \times 10^{-3}\right) dt
\]

Integrating

\[
2 h^{1/2} = -\left(6.266 \times 10^{-3}\right) t + C \quad (\dagger)
\]

Substituting $t = 0, \ h = 1.6$

\[
2 \times 1.6^{1/2} = 0 + C \quad \text{which gives} \quad C = 2.53
\]

Substituting for $C$ into $(\dagger)$ and dividing both sides by 2

\[
h^{1/2} = \frac{-\left(6.266 \times 10^{-3}\right) t + 2.53}{2} = \left(3.133 \times 10^{-3}\right) t + 1.265
\]

Squaring both sides

\[
h = \left(-\left(3.133 \times 10^{-3}\right) t + 1.265\right)^2
\]

(iii) The tank is empty when $h = 0$. Hence

\[
h = \left[-\left(3.133 \times 10^{-3}\right) t + 1.265\right]^2 = 0 \quad \text{gives} \quad t = \frac{1.265}{3.133 \times 10^{-3}} = 403.8
\]

The time taken in minutes to empty the tank is $t = \frac{403.8}{60} = 6.73\text{mins}$.

(iv) The graph is the same shape as 1(iv) but cuts the axis at 1.6 and the $t$ axis at 403.8. When $t \geq 403.8$ the tank is empty, $h = 0$.
3. (i) We have to use similarity of triangles to obtain a relationship between height, \( h \), and \( x \) as shown.

![Diagram of triangles ABC and ADE]

Triangles ABC and ADE are similar so using hint we have
\[
\frac{x}{h} = \frac{1.5}{3.75} = 0.4 \quad \text{transposing gives} \quad x = 0.4h
\]

The cross-sectional area, \( A \), is a circle with radius \( x = 0.4h \)
\[
A = \pi (0.4h)^2 = 0.16\pi h^2
\]

Outlet area, \( A_0 \), is a circle
\[
A_0 = \pi \left( \frac{0.05}{2} \right)^2 = (6.25 \times 10^{-4})\pi
\]

Substituting these into (13.10) and using \( g = 9.81 \) we have
\[
\frac{dh}{dt} = \frac{-\left(6.25 \times 10^{-4}\right)\pi}{0.16\pi h^2} \sqrt{2gh} = \frac{-\left(6.25 \times 10^{-4}\right)}{0.16} \sqrt{2g \frac{\sqrt{h}}{h^2}} = -\left(17.3 \times 10^{-3}\right)h^{-3/2}
\]

(ii) Separating variables
\[
h^{3/2} dh = -\left(17.3 \times 10^{-3}\right) dt
\]
\[
\frac{2h^{5/2}}{5} = -\left(17.3 \times 10^{-3}\right) t + C
\]

Substituting the initial condition \( t = 0, \ h = 3 \) gives
\[
C = \left(2 \times 3^{5/2}\right) \sqrt{5} = 6.235
\]

\[
\frac{2h^{5/2}}{5} = -\left(17.3 \times 10^{-3}\right) t + 6.235
\]

Transposing to make \( h \) the subject gives
\[
h^{5/2} = \frac{5}{2} \left[-\left(17.3 \times 10^{-3}\right) t + 6.235\right] = -\left(43.25 \times 10^{-3}\right) t + 15.59
\]

\[
h = \left[-\left(43.25 \times 10^{-3}\right) t + 15.59\right]^{2/5}
\]

(iii) We need to find \( t \) for \( h = 0 \)
\[
0 = -\left(17.3 \times 10^{-3}\right) t + 6.235 \quad \text{transposing} \quad t = \frac{6.235}{17.3 \times 10^{-3}} = 360
\]

In minutes \( t = 6 \) mins. Takes 6 mins to empty the tank which is 3m full.

(iv) By using a graphical calculator or a symbolic manipulator we have

(13.10)
\[
\frac{dh}{dt} = -\frac{A_0}{A} \sqrt{2gh}
\]
The graph indicates that the height of water in the tank decreases slowly at the start and decreases more rapidly towards the end as you would expect from a conical tank.

(v) Impossible because the tank only has a height of 3.75m.

4. We have

\[
\frac{dh}{dt} = -k\sqrt{2gh} = -k\sqrt{2gh}h
\]

Separating variables

\[
\frac{dh}{h^{1/2}} = -\left(k\sqrt{2g}\right)dt
\]

\[
\int h^{-1/2}dh = -\int\left(k\sqrt{2g}\right)dt
\]

\[
2h^{1/2} = -\left(k\sqrt{2g}\right)t + C
\]

Substituting the initial condition \( t = 0, h = 1 \) gives \( C = 2 \). Thus we have

\[
h^{1/2} = -\frac{1}{2}\left(k\sqrt{2g}\right)t + \frac{2}{2} = -\left(k\sqrt{\frac{2g}{4}}\right)t + 1
\]

Squaring both sides

\[
h = \left[-\left(k\sqrt{\frac{g}{2}}\right)t + 1\right]^2
\]

The graph of height, \( h \), against time, \( t \), for \( k = 0.1, 0.01 \) and \( 0.001 \) is

The graphs show that the height, \( h \), of water in the tank decreases more rapidly for large \( k \).
5. Similar to **EXAMPLE 14**. Substituting \( T = 300 \) into (13.11) we have

\[
\frac{d\theta}{dt} = k(\theta - 300)
\]

Separating variables

\[
\frac{d\theta}{\theta - 300} = kdt
\]

Integrating both sides \( \int \frac{d\theta}{\theta - 300} = \int kdt \) gives

\[
\ln(\theta - 300) = kt + C \quad (\dagger)
\]

Substituting \( t = 0, \ \theta = 373 \) into \((\dagger)\)

\[
\ln(373 - 300) = C \quad \text{gives} \quad C = \ln(73)
\]

Putting this into \((\dagger)\)

\[
\ln(\theta - 300) = kt + \ln(73)
\]

\[
k = \ln(\theta - 300) - \ln(73) = \ln\left(\frac{\theta - 300}{73}\right) \quad (\ddagger)
\]

When \( t = 5 \times 60 = 300, \ \theta = 330 \), substituting these into \((\ddagger)\)

\[
300k = \ln\left(\frac{330 - 300}{73}\right)
\]

\[
k = \frac{\ln(30/73)}{300} = -2.96 \times 10^{-3}
\]

Putting \( k = -2.96 \times 10^{-3} \) into \((\ddagger)\) gives

\[
\ln\left(\frac{\theta - 300}{73}\right) = -(2.96 \times 10^{-3})t
\]

Taking exponentials

\[
\frac{\theta - 300}{73} = e^{-2.96 \times 10^{-3}t} \quad \text{rearranging} \quad \theta = 300 + 73e^{-2.96 \times 10^{-3}t}
\]

For graph: At \( t = 0, \ \theta = 300 + 73e^0 = 373 \) and as \( t \to \infty, \ \theta \to 300 \)

6. From (13.11) we have \( \frac{d\theta}{dt} = k(\theta - 300) \)

because the surrounding temperature is 300K. Separating variables

\[
\frac{d\theta}{\theta - 300} = kdt \quad \text{and integrating} \quad \ln(\theta - 300) = kt + C
\]

Using \( t = 0, \ \theta = 400 \) gives \( C = \ln(100). \) We have

\[
\ln(\theta - 300) - \ln(100) = kt \quad \text{which gives} \quad \ln\left(\frac{\theta - 300}{100}\right) = kt
\]

(13.11) \[
\frac{d\theta}{dt} = k(\theta - T)
\]
Taking exponentials and rearranging gives $\theta = 300 + 100e^{kt}$. We cannot find the value of $k$ in this case because we do not have enough information.

7. Separating variables $\frac{d\theta}{\theta - T} = k dt$. Integrating both sides

$$\int \frac{d\theta}{\theta - T} = \int k dt \text{ gives } \ln(\theta - T) = kt + C \quad (*)$$

Substituting the initial condition $t = 0$, $\theta = T_0$ gives

$$\ln(T_0 - T) = C$$

We have

$$\ln(\theta - T) = kt + \ln(T_0 - T)$$

$$\ln(\theta - T) - \ln(T_0 - T) = kt$$

By using the properties of logs

$$\ln \left( \frac{\theta - T}{T_0 - T} \right) = kt$$

Taking exponentials of both sides

$$\frac{\theta - T}{T_0 - T} = e^{kt}, \text{ rearranging } \theta - T = (T_0 - T)e^{kt}$$

Thus $\theta = T + (T_0 - T)e^{kt}$

8. Using the result of question 7, $\theta = T + (T_0 - T)e^{kt}$, with $T_0 = 348$, $T = 320$ we have $\theta = 320 + 28e^{kt}$. Plotting graphs for $k = -0.001$, $-0.01$ and $-0.1$ gives

The graph shows the larger the absolute value of $k$ the more rapidly the temperature drops to the surrounding temperature of $320 K$.

9. Separating variables $\frac{d\theta}{\theta^4 - T^4} = k dt$. To find $\theta$ we have to integrate the left hand side by using partial fractions.

$$\theta^4 - T^4 = (\theta^2 - T^2)(\theta^2 + T^2) = (\theta - T)(\theta + T)(\theta^2 + T^2)$$

We have

$$\frac{1}{\theta^4 - T^4} = \frac{1}{(\theta - T)(\theta + T)(\theta^2 + T^2)} = \frac{A}{\theta - T} + \frac{B}{\theta + T} + \frac{C\theta + D}{\theta^2 + T^2} \quad (*)$$

Thus

$$1 = A(\theta + T)(\theta^2 + T^2) + B(\theta - T)(\theta^2 + T^2) + (C\theta + D)(\theta - T)(\theta + T) \quad (**)$$

Substituting $\theta = -T$ into (**) gives
Solutions 13(d)  

\[
1 = 0 + B(-T - T)((-T)^2 + T^2) + 0 = B(-2T)(2T^2) \text{ which gives } B = -\frac{1}{4T^3}
\]

How can we find \(A\)?

Substitute \(\theta = T\) into (**)

\[
1 = A(T + T)(T^2 + T^2) + 0 + 0 = 4T^3A \text{ which gives } A = \frac{1}{4T^3}
\]

To find \(C\) we equate coefficients of \(\theta^3\) in (**):

\[
0 = A + B + C \quad \Rightarrow \quad 0 = \frac{1}{4T^3} - \frac{1}{4T^3} + C \text{ gives } C = 0
\]

To find \(D\) we equate coefficients of \(\theta^2\) in (**):

\[
0 = AT - BT + D \Rightarrow \frac{T}{4T^3} + \frac{T}{4T^3} + D = \frac{1}{4T^2} + \frac{1}{4T^2} + D = \frac{1}{2T^2} + D \text{ thus } D = -\frac{1}{2T^2}
\]

Substituting \(A = \frac{1}{4T^3}, \ B = -\frac{1}{4T^3}, \ C = 0\) and \(D = -\frac{1}{2T^2}\) into (*)

\[
\frac{1}{\theta^3 - T^3} = \frac{1}{4T^3} \left[ \frac{1}{\theta - T} - \frac{1}{\theta + T} - \frac{2T}{\theta^2 + T^2} \right]
\]

\[
\int \frac{d\theta}{\theta^4 - T^4} = \frac{1}{4T^3} \left[ \int \frac{d\theta}{\theta - T} - \int \frac{d\theta}{\theta + T} - 2T \int \frac{d\theta}{\theta^2 + T^2} \right] = \frac{1}{4T^3} \left[ \ln(\theta - T) - \ln(\theta + T) - 2T \tan^{-1}\left(\frac{\theta}{T}\right) \right] \quad \text{by (8.26)}
\]

\[
= \frac{1}{4T^3} \left[ \ln\left(\frac{\theta - T}{\theta + T}\right) - 2\tan^{-1}\left(\frac{\theta}{T}\right) \right]
\]

The constant of integration is added at the end on the right hand side. We have \(\int \frac{d\theta}{\theta^4 - T^4} = \int k dt = kt + C\), thus the required result

\[
\frac{1}{4T^3} \left[ \ln\left(\frac{\theta - T}{\theta + T}\right) - 2\tan^{-1}\left(\frac{\theta}{T}\right) \right] = kt + C
\]

\[
(8.26) \quad \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)
\]