Complete solutions to Exercise 16(e)

1. Let $X$ be the number of resistor out of tolerance.
   (a) Need to find $P(X = 3)$ in the binomial distribution, (16.24), with $p = 0.1$, $q = 0.9$ and $n = 5$.
   
   $$P(X = 3) = \binom{5}{3} (0.1)^3 (0.9)^2 = 8.1 \times 10^{-3}$$

   (b) Need to find
   
   $$P(X \leq 3) = 1 - P(X > 3)$$
   
   $$= 1 - \left[ P(X = 4) + P(X = 5) \right] \quad (\dagger)$$

   By (16.24) we have
   
   $$P(X = 4) = \binom{5}{4} (0.1)^4 (0.9) = 4.5 \times 10^{-4}$$

   $$P(X = 5) = 0.1^5 = 1 \times 10^{-5}$$

   Substituting $P(X = 4) = 4.5 \times 10^{-4}$ and $P(X = 5) = 1 \times 10^{-5}$ into $(\dagger)$ gives
   
   $$P(X \leq 3) = 1 - \left[ (4.5 \times 10^{-4}) + (1 \times 10^{-5}) \right] = 0.99954$$

2. Let $X$ be the number of components that fail.
   (a) Using (16.24) with $p = \frac{1}{25} = 0.04$, $q = 1 - 0.04 = 0.96$, $n = 5$ and $x = 1$ gives
   
   $$P(X = 1) = \binom{5}{1} (0.04 \times 0.96)^4 = 0.170 \quad (3 \text{ s.f.})$$

   (b) In this case we have $X > 3$.
   
   $$P(X > 3) = P(X = 4) + P(X = 5)$$
   
   $$= \binom{5}{4} (0.04)^4 (0.96) + (0.04)^5$$

   $$= 1.239 \times 10^{-5} \quad (4 \text{ s.f.})$$

3. Let $X$ be the number of integrated circuits that fail. Using (16.24) with $p = 1 - 0.86 = 0.14$, $q = 0.86$ and $n = 10$ ($p \neq 0.86$ because $X$ is the number of failures).
   (i) We have
   
   $$P(2 \text{ will fail}) = P(X = 2)$$
   
   $$= \binom{10}{2} (0.14)^2 (0.86)^8 = 0.264 \quad (3 \text{ d.p.})$$

   (ii) The probability of 8 will pass is the same as the probability of 2 failing, which is 0.264

   (iii)
   
   $$P(\text{at least 2 will fail}) = 1 - P(X \leq 1)$$
   
   $$= 1 - \left[ P(X = 0) + P(X = 1) \right]$$

   $$= 1 - \left[ 0.86^{10} + \binom{10}{1} (0.14)(0.86)^9 \right]$$

   $$= 1 - 0.5816 = 0.418 \quad (3 \text{ d.p.})$$

(16.24)  

$$P(X = x) = \binom{n}{x} p^x q^{n-x}$$
(iv) We have

\[ P(\text{less than 2 will fail}) = P(X = 0) + P(X = 1) \]

\[ = 0.582 \text{ (3 d.p.)} \]

4. Let \( X \) be the number of boys.

\[ P(\text{having more boys}) = P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7) \]

Using (16.24) with \( n = 7, \ p = 0.48 \) and \( q = 1 - 0.48 = 0.52 \) we have

\[ P(\text{having more boys}) = \binom{7}{4}(0.48)^4(0.52)^3 + \binom{7}{5}(0.48)^5(0.52)^2 + \binom{7}{6}(0.48)^6(0.52) + (0.48)^7 \]

\[ = 0.456 \text{ (3 d.p.)} \]

5. We have

\[ P(\text{passes more than 6 subjects}) = P(\text{passes 7 subjects}) + P(\text{passes 8 subjects}) \]

Binomial distribution with \( p = 0.85, \ q = 0.15 \) and \( n = 8 \)

\[ P(\text{passes 7 subjects}) = \binom{8}{7}(0.85)^7(0.15) = 0.3847 \]

\[ P(\text{passes 8 subjects}) = 0.85^8 = 0.2725 \]

Hence

\[ P(\text{passes more than 6 subjects}) = 0.3847 + 0.2725 = 0.6572 \text{ (4 d.p.)} \]

6. (a) Using (16.24) with \( n = 600, \ p = 1 \times 10^{-3} \) and \( q = 1 - (1 \times 10^{-3}) = 0.999 \). (All solutions correct to 4 d.p.)

(i) \( P(X = 1) = \binom{600}{1}(1 \times 10^{-3})(0.999)^{599} = 0.3295 \)

(ii) \( P(X = 2) = \binom{600}{2}(1 \times 10^{-3})^2(0.999)^{598} = 0.0988 \)

(iii) \( P(X = 3) = \binom{600}{3}(1 \times 10^{-3})^3(0.999)^{597} = 0.0197 \)

(b) We have \( nP = 600 \times 1 \times 10^{-3} = 0.6 \). Using \( P(X = x) = \frac{e^{-0.6}(0.6)^x}{x!} \) gives

(i) \( P(X = 1) = \frac{e^{-0.6}(0.6)^1}{1!} = 0.3293 \)

(ii) \( P(X = 2) = \frac{e^{-0.6}(0.6)^2}{2!} = 0.0988 \)

(iii) \( P(X = 3) = \frac{e^{-0.6}(0.6)^3}{3!} = 0.0198 \)

The answers are more or less the same.

7. Let \( X \) represent England player scores a penalty. Then \( p = 0.85, \ q = 0.15 \) and \( n = 5 \). We need to find \( P(X = 4) + P(X = 5) \)

\[ P(X = 4) + P(X = 5) = \binom{5}{4}(0.85)^4(0.15) + (0.85)^5 \]

\[ = 0.83521 \]

8. Substituting \( x = 0, 1 \) and \( 2 \) into \( P(X = x) = kx^4 \) gives

\[ (16.24) \quad P(X = x) = nC_x p^x q^{n-x} \]
\[ P(X = 0) = 0 \]
\[ P(X = 1) = k \]
\[ P(X = 2) = 2^4 k = 16k \]

By (16.23)
\[ 16k + k = 1 \quad \text{which gives} \quad k = \frac{1}{17} \]

9. Similarly
\[ P(X = 0) = k \cdot 0^3 = 0 \]
\[ P(X = 1) = k \]
\[ P(X = 2) = 2^3 k = 8k \]
\[ P(X = 3) = 3^3 k = 27k \]

By (16.23)
\[ 27k + 8k + k = 1 \]
\[ 36k = 1 \quad \text{which gives} \quad k = \frac{1}{36} \]

10. We have
\[
\sum_{x=1}^{n} x P(X = x) = \left[ 0 \times P(X = 0) \right] + \left[ 1 \times P(X = 1) \right] + \left[ 2 \times P(X = 2) \right] + \left[ 3 \times P(X = 3) \right] + \ldots + \left[ n \times P(X = n) \right] \\
\leq 0 + npq^{n-1} + \frac{2n(n-1)}{2!} p^2 q^{n-2} + \frac{3n(n-1)(n-2)}{3!} p^3 q^{n-3} + \ldots + np^n \\
= npq^{n-1} + n(n-1) p^2 q^{n-2} + \frac{n(n-1)(n-2)}{2} p^3 q^{n-3} + \ldots + np^n \\
= np \left[ q^{n-1} + (n-1) p q^{n-2} + \frac{(n-1)(n-2)}{2!} p^2 q^{n-3} + \ldots + p^{n-1} \right] \\
= np \left[ (q + p)^{n-1} \right] \\
= np \quad \text{because} \quad q + p = 1 \quad \text{and} \quad 1^{n-1} = 1
\]

(7.23) \quad (a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!} a^{n-2}b^2 + \ldots + b^n

(16.24) \quad P(X = x) = \frac{n!}{x!(n-x)!} p^x q^{n-x}