Complete solutions to Exercise 16(f)

1. The probability is \( \frac{1}{5} \) for each \( x \) value, hence:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(X = x) )</td>
<td>( \frac{1}{5} )</td>
<td>( \frac{1}{5} )</td>
<td>( \frac{1}{5} )</td>
<td>( \frac{1}{5} )</td>
<td>( \frac{1}{5} )</td>
</tr>
</tbody>
</table>

By (16.26)

\[
E(X) = \left(1 \times \frac{1}{5}\right) + \left(2 \times \frac{1}{5}\right) + \left(3 \times \frac{1}{5}\right) + \left(4 \times \frac{1}{5}\right) + \left(5 \times \frac{1}{5}\right)
\]

\[= 3\]

Notice the symmetrical nature of the probability distribution, therefore the mean \( E(X) = 3 \).

2. By (16.32)

\[
E(X^2) = (1^2 \times 0.15) + (2^2 \times 0.21) + (3^2 \times 0.11) + (4^2 \times 0.36) + (5^2 \times 0.04) + (6^2 \times 0.13)
\]

\[= 13.42\]

3. By using your calculator the mean is 3 and S.D. is 1.095 (3 d.p.). Since the probability distribution is symmetrical the mean is clearly going to be 3.

4. Easiest way to tackle this problem is to use your calculator. The mean \( \mu = 4.19V \) (2 d.p.) and S.D. \( \sigma = 1.94V \) (2 d.p.)

5. The mean should be close to 1V because the signal is concentrated near 1V, that is higher probabilities are close to 1V. Use calculator: mean = 1.11V (2 d.p.) and S.D. = 1.05V (2 d.p.)

6. (i) Substituting the given \( x \) values into \( P(X = x) = kx^2 \) gives

\[
P(X = 1) = k
\]

\[
P(X = 2) = 4k
\]

\[
P(X = 3) = 9k
\]

\[
P(X = 4) = 16k
\]

By (16.23), \( 16k + 9k + 4k + k = 1 \)

\[30k = 1 \text{ which gives } k = \frac{1}{30}\]

Substituting \( k = \frac{1}{30} \) and the \( x \) values into \( P(X = x) = kx^2 \) gives

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(X = x) )</td>
<td>( \frac{1}{30} )</td>
<td>( \frac{4}{30} )</td>
<td>( \frac{9}{30} )</td>
<td>( \frac{16}{30} )</td>
</tr>
</tbody>
</table>

By using a calculator we have:

(ii) \( E(X) = 3.33 \) (2 d.p.)

(iii) \( S.D. = 0.83 \) (2.d.p.)

(iv) variance = \( 0.83^2 = 0.69 \) (2 d.p.)

(16.26) \[E(X) = x_1P(X = x_1) + x_2P(X = x_2) + \ldots + x_nP(X = x_n)\]

(16.32) \[E(X^2) = x_1^2P(X = x_1) + x_2^2P(X = x_2) + \ldots + x_n^2P(X = x_n)\]
7. (i) Use calculator or
\[ E(X) = (1 \times 0.1) + (2 \times 0.3) + (3 \times 0.2) + (4 \times 0.4) = 2.9 \]
(ii)
\[ E(X^2) = (1^2 \times 0.1) + (2^2 \times 0.3) + (3^2 \times 0.2) + (4^2 \times 0.4) = 9.5 \]
(iii)
\[ E(X^2 + X) = \frac{E(X^2) + E(X)}{\text{by (16.29)}} = \frac{9.5 + 2.9}{12.4} = 9.5 + 2.9 = 12.4 \]
(iv)
\[ E(5X^2 + 7X + 3) = \frac{5E(X^2) + 7E(X) + 3}{\text{by (16.29)}} = (5 \times 9.5) + (7 \times 2.9) + 3 = 70.8 \]

8. (i)
\[ E(k) = \sum_{\text{all } x} kP(X = x) = k \text{ } \sum_{\text{all } x} P(X = x) = k \]
= because of (16.23)
(ii)
\[ E[af(X) + b] = \sum_{\text{all } x} [af(X) + b]P(X = x) = \sum_{\text{all } x} [af(X)P(X = x) + bP(X = x)] = \sum_{\text{all } x} af(X)P(X = x) + \sum_{\text{all } x} bP(X = x) = a \sum_{\text{all } x} f(x)P(X = x) + \sum_{\text{all } x} bP(X = x) = aE[f(x)] + E(b) = aE[f(x)] + b \text{ by (i)} \]

(16.23) \[ p_1 + p_2 + \ldots + p_n = 1 \]
(16.29) \[ E[kf(X) + m] = kE[f(x)] + m \]
9. We are given the following probability distribution:

<table>
<thead>
<tr>
<th>$x_j$ (number of goals scored)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X = x_j)$</td>
<td>0.05</td>
<td>0.15</td>
<td>0.2</td>
<td>0.25</td>
<td>0.15</td>
<td>0.1</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Mean number of goals $\mu = E(X)$ can be determined by using formula (16.26):

$\mu = E(X) = x_1P(X = x_1) + x_2P(X = x_2) + \cdots + x_nP(X = x_n) + x_nP(X = x_n)$

$= (0 \times 0.05) + (1 \times 0.15) + (2 \times 0.2) + (3 \times 0.25) + (4 \times 0.15) + (5 \times 0.1) + (6 \times 0.1)$

$= 3$

The mean number of goals is $\mu = 3$.

The standard deviation is the square root of the variance. The variance formula is given by (16.31)

$\sigma^2 = E(X^2) - \mu^2$

Above we found $\mu = 3$ which means we only need to find $E(X^2)$:

$E(X^2) = x_1^2P(X = x_1) + x_2^2P(X = x_2) + \cdots + x_n^2P(X = x_n) + x_n^2P(X = x_n)$

$= (0^2 \times 0.05) + (1^2 \times 0.15) + (2^2 \times 0.2) + (3^2 \times 0.25) + (4^2 \times 0.15) + (5^2 \times 0.1) + (6^2 \times 0.1)$

$= 11.7$

Substituting $E(X^2) = 11.7$ and $\mu = 3$ into the above formula $\sigma^2 = E(X^2) - \mu^2$ gives

$\sigma^2 = 11.7 - 3^2 = 2.7$

The standard deviation is the square root of this $\sigma = \sqrt{2.7} = 1.643$ (3dp).

10. Let $X$ be the number of girls in a family with three children. We are given that

$P(G) = \frac{1}{2}$ and $P(B) = \frac{1}{2}$

This is binomial distribution with $n = 3$:

$(p + q)^3 = p^3 + 3p^2q + 3pq^2 + q^3$ (*)

Let $p$ be the probability of having a girl, then $p = \frac{1}{2}$ and $q = \frac{1}{2}$. Substituting these into (*) yields

$\left(\frac{1}{2} + \frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3$

$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8}$
The probability distribution is given by

<table>
<thead>
<tr>
<th>Number of girls - $x_j$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X = x_j)$</td>
<td>$\frac{1}{8}$</td>
<td>$\frac{3}{8}$</td>
<td>$\frac{3}{8}$</td>
<td>$\frac{1}{8}$</td>
</tr>
</tbody>
</table>

The expected value is given by formula (16.26):

$$E(X) = x_1P(X = x_1) + x_2P(X = x_2) + \cdots + x_nP(X = x_n)$$

$$= \left(0 \times \frac{1}{8}\right) + \left(1 \times \frac{3}{8}\right) + \left(2 \times \frac{3}{8}\right) + \left(3 \times \frac{1}{8}\right) = 1.5$$

This is an example where statistics is not particularly helpful. Clearly we cannot have 1.5 girls in a family.

11. The probability distribution is given by:

<table>
<thead>
<tr>
<th>Score $x_j$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X = x_j)$</td>
<td>$\frac{1}{36}$</td>
<td>$\frac{2}{36}$</td>
<td>$\frac{3}{36}$</td>
<td>$\frac{4}{36}$</td>
<td>$\frac{5}{36}$</td>
<td>$\frac{6}{36}$</td>
<td>$\frac{5}{36}$</td>
<td>$\frac{4}{36}$</td>
<td>$\frac{3}{36}$</td>
<td>$\frac{2}{36}$</td>
<td>$\frac{1}{36}$</td>
</tr>
</tbody>
</table>

Since the table is symmetrical about the score 7 therefore the expected value should be 7.

We can confirm this expected value by using formula (16.26):

$$E(X) = x_1P(X = x_1) + x_2P(X = x_2) + \cdots + x_{10}P(X = x_{10}) + x_{11}P(X = x_{11})$$

$$= \frac{1}{36} \left[ 2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12 \right] = 7$$

The variance is given by the formula $\sigma^2 = E(X^2) - 7^2$. We need to evaluate $E(X^2)$:

$$E(X^2) = x_1^2P(X = x_1) + x_2^2P(X = x_2) + \cdots + x_{10}^2P(X = x_{10}) + x_{11}^2P(X = x_{11})$$

$$= \frac{1}{36} \left[ 2^2 + (3^2 \times 2) + (4^2 \times 3) + (5^2 \times 4) + (6^2 \times 5) + (7^2 \times 6) + (8^2 \times 5) + (9^2 \times 4) + (10^2 \times 3) + (11^2 \times 2) + 12^2 \right]$$

$$= 54.833$$

Substituting $E(X^2) = 54.833$ into $\sigma^2 = E(X^2) - 7^2$ gives

$$\sigma^2 = 54.833 - 7^2 = 5.833 \text{ (3dp)}$$
12. Using (16.33) we have

\[ P(X = x) = \frac{e^{-1.6}1.6^x}{x!} \]

(a) \( P(X = 0) = \frac{e^{-1.6}1.6^0}{0!} = 0.20 \)

(b) \( P(X = 1) = \frac{e^{-1.6}1.6^1}{1!} = 0.32 \)

(c) \[ P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = 0.2 + 0.32 + \frac{e^{-1.6}1.6^2}{2!} = 0.78 \]

(d) \( P(X = 10) = \frac{e^{-1.6}1.6^{10}}{10!} = 6.12 \times 10^{-6} \)

13. The mean is given by

\[ \mu = np = 4096 \times (1 \times 10^{-3}) = 4.096 \]

(a) Using (16.33)

\[ P(X = 0) = \frac{e^{-4.096}(4.096)^0}{0!} = 0.0167 \]

(b) The probability for more than 2 errors is given by

\[ 1 - [P(X = 0) + P(X = 1) + P(X = 2)] = 1 - [0.0167 + 0.068 + 0.14] = 0.776 \]

(16.23) \( p_1 + p_2 + \ldots + p_n = 1 \)

(16.33) \( P(X = x) = e^{-\mu}(\mu)^x/x! \)