Complete solutions to Exercise 2(d)

1. (a) \[ x^2 - 4x + 3 = (x - 2)^2 - 4 + 3 \]
   \[ = (x - 2)^2 - 1 \]

   (b) How do we complete the square on \( x^2 + 8x + 9 \)?
   Consider half the \( x \) coefficient, that is half of 8:
   \[ x^2 + 8x + 9 = (x + 4)^2 - 16 + 9 \]
   \[ = (x + 4)^2 - 7 \]

   (c) Similarly we have
   \[ x^2 - 6x + 8 = (x - 3)^2 - 9 + 8 \]
   \[ = (x - 3)^2 - 1 \]

   (d) Completing the square on the given quadratic we have
   \[ x^2 - 10x + 2 = (x - 5)^2 - 25 + 2 \]
   \[ = (x - 5)^2 - 23 \]

   (e) \[ 9 + 8x - x^2 = 9 - (x^2 - 8x) \]
   \[ = 9 - \left( (x - 4)^2 - \frac{16}{4^2} \right) = 25 - (x - 4)^2 \]

   (f) \[ x^2 + 7x + 1 = \left( x + \frac{7}{2} \right)^2 - \left( \frac{7}{2} \right)^2 + 1 = \left( x + \frac{7}{2} \right)^2 - \frac{45}{4} \]

   (g) \[ 2x^2 + 7x + 1 = 2 \left[ x^2 + \frac{7}{2}x + \frac{1}{2} \right] \]
   \[ = 2 \left[ \left( x + \frac{7}{4} \right)^2 - \left( \frac{7}{4} \right)^2 + \frac{1}{2} \right] \]
   \[ = 2 \left( x + \frac{7}{4} \right)^2 - \frac{41}{16} \]
   \[ = 2 \left( x + \frac{7}{4} \right)^2 - \frac{41}{8} \]

2. Similar to Example 11. All the hard work has been done in solution 1.
(a) From solution 1(a) we have
   \( (x - 2)^2 - 1 = 0 \)
   \[ (x - 2)^2 = 1, \ x - 2 = \pm 1 \] which gives \( x = 1, \ x = 3 \)

(b) By the above solution 1(b):
   \[ x^2 + 8x + 9 = (x + 4)^2 - 7 = 0 \]
   \[ (x + 4)^2 = 7 \] implies that \( x + 4 = \pm \sqrt{7} \)
   This means that \( x = -4 \pm \sqrt{7} \) which is \( x = -4 - \sqrt{7}, \ x = -4 + \sqrt{7} \)

(c) By solution 1(c) we have \( x^2 - 6x + 8 = (x - 3)^2 - 1 \). Equating this to zero yields
\[
(x-3)^2 - 1 = 0 \\
(x-3)^2 = 1 \quad \text{which means that } x - 3 = \pm 1
\]
We have \( x = 3 \pm 1 = 2, 4 \).

(d) From solution to 1(d) we have \( x^2 - 10x + 2 = (x-5)^2 - 23 \). Equating this to zero yields:
\[
(x-5)^2 - 23 = 0 \\
(x-5)^2 = 23 \\
x - 5 = \pm \sqrt{23} \quad \text{implies that } x = 5 \pm \sqrt{23}
\]
We have \( x = 5 - \sqrt{23}, \ x = 5 + \sqrt{23} \).

(e) Similarly
\[
25 - (x - 4)^2 = 0 \\
(x - 4)^2 = 25, \ x - 4 = \pm \sqrt{25} = \pm 5
\]
\( x = 4 \pm 5 \) which gives \( x = -1, 9 \)

(f) We have
\[
\left( x + \frac{7}{2} \right)^2 - \frac{45}{4} = 0 \\
\left( x + \frac{7}{2} \right)^2 = \frac{45}{4}, \ x + \frac{7}{2} = \pm \frac{\sqrt{45}}{2} = \pm \frac{\sqrt{45}}{2}
\]
\( x = \frac{-7 \pm \sqrt{45}}{2} = -\frac{7 \pm \sqrt{45}}{2} \)

(g) Very similar to the above. We have
\[
2\left( x + \frac{7}{4} \right)^2 - \frac{41}{8} = 0 \\
2\left( x + \frac{7}{4} \right)^2 = \frac{41}{8}, \ \left( x + \frac{7}{4} \right)^2 = \frac{41}{16}
\]
\( x + \frac{7}{4} = \pm \frac{\sqrt{41}}{4} = \pm \frac{\sqrt{41}}{4} \)
\( x = \frac{-7 \pm \sqrt{41}}{4} \)

3. When \( t = 0, \ V = 6 \). To find minimum or maximum we need to complete the square.
\[
V = t^2 - 5t + 6 \\
= \left( t - \frac{5}{2} \right)^2 - \left( \frac{5}{2} \right)^2 + 6
\]
\[
V = \left( t - \frac{5}{2} \right)^2 - \frac{1}{4}
\]
We have a minimum at \( t = \frac{5}{2} = 2.5 \) with \( V = -\frac{1}{4} \). Need to find the values of \( t \) which gives \( V = 0 \).
\( t^2 - 5t + 6 = 0 \)
\[ (t - 3)(t - 2) = 0 \]
\[ t = 3, \ 2 \]

Hence the graph \( V \) cuts the \( t \) axis at \( t = 2, \ t = 3 \).

4. At \( t = 0, \ V = 0 \). Completing the square:
\[ V = t^2 - t \]
\[ = \left( t - \frac{1}{2} \right)^2 - \frac{1}{4} \]
This gives a minimum at \( t = \frac{1}{2} = 0.5 \) with \( V = -\frac{1}{4} \). To find the values of \( t \) which gives zero voltage,
\[ t^2 - t = 0 \]
\[ t(t - 1) = 0 \] gives \( t = 0, \ t = 1 \)

Combining all the information gives the graph:
5. Since there is a $- t^2$ term, we know it is a quadratic of the form $\cap$. Also $h = 12t - t^2$ cuts the t axis where $h = 0$:

\[ 12t - t^2 = 0 \]

\[ t(12 - t) = 0 \text{ gives } t = 0, t = 12 \]

The highest point is reached when $12t - t^2$ is a maximum. It can be found by completing the square:

\[ 12t - t^2 = -(t^2 - 12t) \]

\[ = -\left[ (t - 6)^2 - 36 \right] \]

\[ = -36 - (t - 6)^2. \]

Hence $t = 6$ gives maximum height $h = 36$:

6. To find the maximum height we need to complete the square on $h$.

\[ h = 200t - 4.9t^2 = -(4.9t^2 - 200t) \]

\[ = -4.9\left( t^2 - \frac{200}{4.9}t \right) \]

\[ = -4.9\left( t^2 - 40.82t \right) \]

\[ = -4.9\left( t - \frac{40.82}{2} \right)^2 \left[ + 4.9 \times \left( \frac{40.82}{2} \right)^2 \right] \]

\[ h = -4.9(t - 20.41)^2 + 2041.18 \]

The maximum height reached is 2041.18m when $t = 20.41s$. The height, $h = 0$ when

\[ 200t - 4.9t^2 = 0 \]

\[ t(200 - 4.9t) = 0 \]

\[ t = 0, \quad t = \frac{200}{4.9} = 40.82 \]
7. $M = 0$ when $Lx - x^2 = 0$. That is $x(L - x) = 0$ gives $x = 0, x = L$.

To find maximum or minimum, complete the square on $Lx - x^2$;

$$Lx - x^2 = -(x^2 - Lx)$$

$$= -(x - \frac{L}{2})^2 + \frac{L^2}{4}$$

Maximum occurs at $x = \frac{L}{2}$. To find the maximum value we substitute $x = \frac{L}{2}$ into

$$M = \frac{W}{24EI} (Lx - x^2)$$

$$M = \frac{W}{24EI} \left[ L \cdot \frac{L}{2} - \frac{L^2}{4} \right] = \frac{W}{24EI} \left[ \frac{L^2}{2} - \frac{L^2}{4} \right]$$

$$= \frac{W}{24EI} \left( \frac{L^2}{4} \right)$$

Maximum value is

$$M = \frac{WL^2}{(24EI) \times 4} = \frac{WL^2}{96EI}$$

Hence the graph is
8. We divide the quadratic equation \( ax^2 + bx + c = 0 \) by \( a \):

\[
\frac{x^2}{a} + \frac{b}{a}x + \frac{c}{a} = 0
\]

Completing the square:

\[
\left( x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0
\]

\[
\left( x + \frac{b}{2a} \right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}
\]

\[
= \frac{b^2 - 4ac}{4a^2}
\]

Taking the square roots of both sides:

\[
x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}
\]

Making \( x \) the subject:

\[
x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}
\]