1. a) \( \sum_{n=1}^{\infty} \sqrt{n} \)  
   b) \( \sum_{n=1}^{\infty} (2n) \)  
   c) \( \sum_{n=1}^{\infty} (2n-1) \)  
   d) \( \sum_{n=1}^{\infty} \left( \frac{(-1)^{n+1}}{n} \right) \)  
   e) \( \sum_{n=0}^{\infty} \left( \frac{1}{3} \right)^n \)  
   f) \( \sum_{n=1}^{\infty} \left( \frac{2}{3} \right)^n \)

2. Each of these is a geometric series with the common ratio \( r \) less than 1 so we can use the following formula to find the sum:

\[
\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} = \frac{\text{First term}}{1-\text{Common ratio}}
\]

a) For the series \( \sum_{n=1}^{\infty} \left( \frac{1}{3} \right)^n \) the first term is \( a = \frac{1}{3} \) because we start with index 1, that is \( \left( \frac{1}{3} \right)^1 = \frac{1}{3} \). Common ratio \( r = \frac{1}{3} \). Substituting these into the above formula (7.27) gives

\[
\sum_{n=0}^{\infty} \left( \frac{1}{3} \right)^n = \frac{1/3}{1-1/3} = \frac{1/3}{2/3} = \frac{1}{2} \quad \text{[Multiplying numerator and denominator by 3]}
\]

b) Similarly for \( \sum_{n=1}^{\infty} \left( \frac{1}{4} \right)^n \) we have \( a = r = \frac{1}{4} \). Substituting these into the above formula

\[
\sum_{n=1}^{\infty} \left( \frac{1}{4} \right)^n = \frac{1/4}{1-1/4} = \frac{1/4}{3/4} = \frac{1}{3} \quad \text{[Multiplying numerator and denominator by 4]}
\]

c) Very similar to parts (a) and (b), we have

\[
\sum_{n=1}^{\infty} \left( \frac{1}{\pi} \right)^n = \frac{\pi}{1-1/\pi} = \frac{\pi}{\pi-1} = \frac{1}{\pi-1} \quad \text{[Multiplying numerator and denominator by \( \pi \)]}
\]

d) This time we have \( m > 1 \) therefore the common ratio \( \frac{1}{m} < 1 \) so we can apply the above formula to find the sum of the infinite series:

\[
\sum_{n=1}^{\infty} \left( \frac{1}{m} \right)^n = \frac{1/m}{1-1/m} = \frac{1/m}{(m-1)/m} = \frac{1}{m-1} \quad \text{[Multiplying numerator and denominator by \( m \)]}
\]
3. a) We have
\[ \sum_{n=1}^{\infty} \left( \frac{1}{2^{2n-1}} \right) = \frac{1}{2^{(2 \times 1)-1}} + \frac{1}{2^{(2 \times 2)-1}} + \frac{1}{2^{(2 \times 3)-1}} + \frac{1}{2^{(2 \times 4)-1}} + \cdots \]
\[ = \frac{1}{2^1} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + \cdots \]

What is the common ratio in this case?

Common ratio \( r = \frac{1}{2^2} = \frac{1}{4} \) which is less than 1 so the series converges and we can use the formula

\( (7.27) \)
\[ \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} = \frac{\text{First term}}{1-\text{Common ratio}} \]

What is the first term \( a \) equal to?

\( a = \frac{1}{2} \). Substituting \( a = \frac{1}{2} \) and \( r = \frac{1}{4} \) into the above formula (7.27) we have

\[ \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-1/4} \]
\[ = \frac{1/2}{3/4} = \frac{2}{3} \]

[Multiplying numerator and denominator by 4]

b) Diverges because we are given \( \sum_{n=1}^{\infty} \left( \frac{3}{2} \right)^n \) which is

\[ \sum_{n=1}^{\infty} \left( \frac{3}{2} \right)^n = \frac{3}{2} + \left( \frac{3}{2} \right)^2 + \left( \frac{3}{2} \right)^3 + \left( \frac{3}{2} \right)^4 + \cdots \]

What is the common ratio \( r \) equal to?

\( r = \frac{3}{2} \) which is greater than 1

Hence the series diverges.

c) Diverges because we have \( \sum_{n=1}^{\infty} (e)^n \) and the common ratio

\( r = e = 2.71828 \cdots \)

The common ratio is greater than 1 so the series diverges.

d) Does the given series \( \sum_{n=1}^{\infty} \left( \frac{1}{3} \right)^n \) converge or not?

Writing out this series we have

\[ \sum_{n=1}^{\infty} \left( \frac{1}{3} \right)^n = 10 \left( \frac{1}{3} \right)^1 + 10 \left( \frac{1}{3} \right)^2 + 10 \left( \frac{1}{3} \right)^3 + 10 \left( \frac{1}{3} \right)^4 + \cdots \]
\[ \equiv 10 \left[ \left( \frac{1}{3} \right) + \left( \frac{1}{3} \right)^2 + \left( \frac{1}{3} \right)^3 + \left( \frac{1}{3} \right)^4 + \cdots \right] \]

Taking out a common factor of 10

The common ratio is \( \frac{1}{3} \) which is less than 1 so the given series converges and
\[ \sum_{n=1}^{\infty} 10 \left( \frac{1}{3} \right)^n = \frac{10}{3} / 1 - 1/3 \]
\[ = \frac{10}{3} \times 2/3 = \frac{10}{2} = 5 \]

4. The total distance \( D \) travelled by the ball is given by the infinite series:
\[ D = 10 + (10 \times 0.55) + \left(10 \times 0.55^2\right) + \left(10 \times 0.55^3\right) + \cdots \]
\[ = \sum_{n=0}^{\infty} 10(0.55)^n \]

**How can we find \( D \)?**

\( D \) is a geometric series with first term \( a = 10 \) and common ratio \( r = 0.55 \). Since \( |r| = 0.55 \) is less than 1 therefore the sum of this series is given by
\[ D = \frac{10}{1 - 0.55} = 22.22 \text{ m} \]

Hence the total distance travelled by the ball is 22.22m (2dp).

5. The maximum rise \( R \) of the balloon is given by:
\[ R = 50 + (50 \times 0.65) + \left(50 \times 0.65^2\right) + \left(50 \times 0.65^3\right) + \cdots = \sum_{n=0}^{\infty} 50(0.65)^n \]

**How can we find \( R \)?**

\( R \) is a geometric series with first term \( a = 50 \) and common ratio \( r = 0.65 \). Since \( |r| = 0.65 \) is less than 1 therefore the sum of this series is given by
\[ D = \frac{50}{1 - 0.65} = 142.86 \text{ m} \]

Hence the maximum rise by the balloon is 142.86m (2dp).

6. We are given that the area removed is \( A = \sum_{n=0}^{\infty} \left( \frac{1}{4} \right) \left( \frac{3}{4} \right)^n \). Writing this out we have:
\[ A = \sum_{n=0}^{\infty} \left( \frac{1}{4} \right) \left( \frac{3}{4} \right)^n = \frac{1}{4} + \frac{1}{4} \left( \frac{3}{4} \right) + \frac{1}{4} \left( \frac{3}{4} \right)^2 + \cdots \]

**How can we find \( A \)?**

\( A \) is a geometric series with first term \( a = \frac{1}{4} \) and common ratio \( r = \frac{3}{4} \). Since \( |r| \) is less than 1 therefore we can find the sum of this infinite series. We have
\[ A = \frac{\frac{1}{4}}{1 - \frac{3}{4}} = \frac{1}{1} \]

\( A = 1 \) means that the whole area is removed.

7. (a) Is \( S = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots \) a geometric series?
Yes because each of term is 1/10 of the previous term. What is the sum of this series?
Since the common ratio $r = \frac{1}{10}$ which means that the modulus of this is less than 1
therefore the sum of the given infinite series is

$$S = \frac{9/10}{1-1/10} = 1$$

This means that $S = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots = 0.999\cdots = 1$.

(b) Similarly we are given that $S = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots$ which is a geometric series with
the same common ratio of $r = \frac{1}{10}$ and first term $a = \frac{9}{10}$ which means we can find the sum
of this infinite series.

$$S = \frac{3/10}{1-1/10} = \frac{1}{3}$$

As part (a) this means that $S = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots = 0.333\cdots = \frac{1}{3}$.

(c) A carbon copy of the solutions presented in parts (a) and (b) gives that the sum of

$$S = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots$$

with $a = 1/10$, $r = 1/10$ is

$$S = \frac{1/10}{1-1/10} = \frac{1}{9}$$

We conclude that this means $S = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots = 0.111\cdots = \frac{1}{9}$.

Parts (a), (b) and (c) show that $0.999\cdots = 1$, $0.333\cdots = \frac{1}{3}$ and $0.111\cdots = \frac{1}{9}$.

8. The total profit $P$ is given by

$$P = 100 + 0.91(100) + 0.91^2(100) + 0.91^3(100) + \cdots$$

This is a geometric series with first term $a = 100$ and common ratio $r = 0.91$. Since the
common ratio is $|r| = 0.91$ and $|r| < 1$ therefore the series converges. We have

$$P = \frac{100}{1-0.91} = 1111.11$$

The total possible profit is £1111.11 (2dp).

9. (a) We need to test the given series $8 + 4 + 2 + 1 + \cdots$ for convergence. How can we write this series in compact form?
8 + 4 + 2 + 1 + ... = 8 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \frac{1}{2} + \left(\frac{1}{2}\right)^3 + ... \\
= 8 \left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + ...ight) \\
= 8 \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \\

The series \(8 \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n\) is a geometric series. Does this series converge?

The common ratio is \(\frac{1}{2}\) so the series converges and we use the formula

\[
\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} = \frac{\text{First term}}{1 - \text{Common ratio}}
\]

to find the sum of the infinite series. What is the first term in this case?

Clearly it is 8. Hence substituting \(a = 8\) and \(r = \frac{1}{2}\) into (7.27) gives

\[
8 \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{8}{1 - \frac{1}{2}} = \frac{8}{\frac{1}{2}} = 16
\]

The sum of the infinite series is 16.

(b) The given series \(3 + 6 + 12 + 24 + ...\) diverges. Why?

By (7.25) we have \(\lim_{n \to \infty} (a_n) \neq 0\) then \(\sum (a_n)\) diverges. This means that if the nth term does not tend towards zero then the series diverges. Since our series \(3 + 6 + 12 + 24 + ...\) gets bigger so it diverges.

(c) For the given series \(16 + 12 + 9 + \frac{27}{4} + ...\) it is difficult to write down a formula in compact form. However we can divide two consecutive terms:

\[
\frac{12}{16} = \frac{9}{12} = \frac{27/4}{9} = \ldots = \frac{3}{4}
\]

This means we have a geometric series with a common ratio of \(\frac{3}{4}\). The first term is 16 and so the sum of the infinite series is

\[
16 + 12 + 9 + \frac{27}{4} + ... = \frac{\text{First term}}{1 - \text{Common ratio}} = \frac{16}{1 - \frac{3}{4}} = \frac{16}{1/4} = 64
\]

10. (a) We are given the series:

\[
\sum_{n=1}^{\infty} \frac{1}{x^n} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + ... 
\]

What is the common ratio \(r\) equal to?

\(r = \frac{1}{x}\). Since \(|x| > 1\) which means that \(|r| = \left|\frac{1}{x}\right| < 1\) so the series converges and the sum is
\[ \sum_{n=1}^{\infty} \frac{1}{x^n} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \cdots \]

First term \[ \frac{1}{1 - \text{Common ratio}} \]

\[ = \frac{1}{x - 1} \quad \text{Multiplying numerator and denominator by } x \]

(b) We are given the series \[ \sum_{n=1}^{\infty} \frac{x^n}{2^n} = \sum_{n=1}^{\infty} \left( \frac{x}{2} \right)^n = \frac{x}{2} + \left( \frac{x}{2} \right)^2 + \left( \frac{x}{2} \right)^3 + \left( \frac{x}{2} \right)^4 + \cdots \]

We have \(|x| < 2\) therefore \(|r| = \left| \frac{x}{2} \right| < 1\) which means that the series converges. Using (7.27) \[ \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} = \frac{\text{First term}}{1 - \text{Common ratio}} \]

we have \[ \sum_{n=1}^{\infty} \left( \frac{x}{2} \right)^n = \frac{\text{First term}}{1 - \text{Common ratio}} \]

\[ = \frac{x/2}{1 - x/2} = \frac{x}{2 - x} \quad \text{Multiplying numerator and denominator by } 2 \]

(c) Similarly we have \[ \sum_{n=1}^{\infty} \frac{1}{(1+x)^n} = \frac{1}{1+x} + \frac{1}{(1+x)^2} + \frac{1}{(1+x)^3} + \frac{1}{(1+x)^4} + \cdots \]

What is the common ratio \(r\) and first term \(a\) equal to in this case?

\[ r = a = \frac{1}{1+x} \]

Since we are given that \(x > 0\) so \(r = \frac{1}{1+x} < 1\). Hence the series converges and \[ \sum_{n=1}^{\infty} \frac{1}{(1+x)^n} = \frac{\text{First term}}{1 - \text{Common ratio}} \]

\[ = \frac{1/(x+1)}{1 - \frac{1}{x+1}} = \frac{1}{x} \quad \text{Multiplying numerator and denominator by } x+1 \]

(d) The series given in this part is very similar to the one in part (c) above. We have \[ \sum_{n=1}^{\infty} \frac{1}{(1+x^2)^n} = \frac{1}{1+x^2} + \frac{1}{(1+x^2)^2} + \frac{1}{(1+x^2)^3} + \frac{1}{(1+x^2)^4} + \cdots \]

Also \(r = a = \frac{1}{1+x^2}\). We are given \(x \neq 0\) so \(r = \frac{1}{1+x^2} < 1\). Hence
\[
\sum_{n=0}^{\infty} \frac{1}{(1+x^2)^n} = \frac{a}{1-r}
\]
\[
= \frac{1/(1+x^2)}{1 - 1/(1+x^2)} = \frac{1}{x^2} \quad \left[ \text{Multiplying numerator and denominator by } 1+x^2 \right]
\]

11. In many of these cases we apply the ratio test which is \( \lim_{n \to \infty} \left( \frac{a_{n+1}}{a_n} \right) = L \). The series only converges if \( L < 1 \).

(a) We are given \( \sum \left( \frac{1}{(2n)!} \right) \). This means that \( a_n = \frac{1}{(2n)!} \) and so the next term \( n+1 \) is

\[
a_{n+1} = \frac{1}{(2(n+1))!} = \frac{1}{(2n+2)!}
\]

Substituting these \( a_n = \frac{1}{(2n)!} \) and \( a_{n+1} = \frac{1}{(2n+2)!} \) into \( L = \lim_{n \to \infty} \left( \frac{a_{n+1}}{a_n} \right) \) gives

\[
L = \lim_{n \to \infty} \left( \frac{a_{n+1}}{a_n} \right) = \lim_{n \to \infty} \left[ \frac{1}{(2n+2)!} \div \frac{1}{(2n)!} \right] = \lim_{n \to \infty} \left[ \frac{1}{(2n+2)!} \times \frac{(2n)!}{1} \right] = \lim_{n \to \infty} \left[ \frac{(2n)!}{(2n+2)!} \right] = \lim_{n \to \infty} \left[ \frac{1}{(2n+2)(2n+1)} \right] = 0
\]

Since \( L = 0 \) the series converges.

(b) We need to evaluate \( L \) to test for convergence. For \( \sum_{n=0}^{\infty} \left( \frac{n!}{2^n} \right) \) we have \( a_n = \frac{n!}{2^n} \) and so

\[
a_{n+1} = \frac{(n+1)!}{2^{n+1}}.
\]

Substituting these into \( L = \lim_{n \to \infty} \left( \frac{a_{n+1}}{a_n} \right) \) gives

\[
L = \lim_{n \to \infty} \left( \frac{a_{n+1}}{a_n} \right) = \lim_{n \to \infty} \left( \frac{(n+1)!}{2^{n+1}} \div \frac{n!}{2^n} \right) = \lim_{n \to \infty} \left( \frac{(n+1)!}{2^{n+1}} \times \frac{2^n}{n!} \right) = \lim_{n \to \infty} \left( \frac{n+1}{2} \right) = \infty
\]

Since \( L = +\infty \) the given series diverges.

(c) Very similar to part (b) with the 2 being replaced by 3. We find that \( L = +\infty \) so the given series diverges.
(d) We have the series \( \sum \left( \frac{(n+1)^2}{2^n} \right) \) and we need to determine \( L = \lim_{n \to \infty} \left( \frac{a_{n+1}}{a_n} \right) \). In this case

\[
a_n = \frac{(n+1)^2}{2^n} \quad \text{and replacing } n \text{ with } n+1 \text{ yields } a_{n+1} = \frac{(n+2)^2}{2^{n+1}}
\]

\[
L = \lim_{n \to \infty} \left( \frac{a_{n+1}}{a_n} \right)
\]

\[
= \lim_{n \to \infty} \left[ \frac{(n+2)^2}{2^{n+1}} \div \frac{(n+1)^2}{2^n} \right]
\]

\[
= \lim_{n \to \infty} \left[ \frac{(n+2)^2}{2^{n+1}} \times \frac{2^n}{(n+1)^2} \right] \quad \text{[Inverting the second fraction and multiplying]}
\]

\[
= \lim_{n \to \infty} \left[ \frac{1}{2} \left( \frac{n+2}{n+1} \right)^2 \right]
\]

\[
= \lim_{n \to \infty} \left[ \frac{1}{2} \left( \frac{n+1+1}{n+1} \right)^2 \right] = \lim_{n \to \infty} \left[ \frac{1}{2} \left( 1 + \frac{1}{n+1} \right)^2 \right] = \lim_{n \to \infty} \left[ \frac{1}{2} \left( 1 + 0 \right)^2 \right] = \frac{1}{2}
\]

Since \( L = \frac{1}{2} < 1 \) the series converges.

(e) We are given the series \( \sum (e^{-n}) \) which can be written in expanded form as

\[
\sum (e^{-n}) = e^{-1} + e^{-2} + e^{-3} + e^{-4} + \cdots = \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \frac{1}{e^4} + \cdots
\]

This is a geometric series with \( r = a = \frac{1}{e} = \frac{1}{2.71828\ldots} \). Since \( r < 1 \) so the series converges.

By using the ratio test we get \( L = \frac{1}{e} \). (The sum is \( \frac{1}{e-1} \).)

(f) We have \( \sum \left( \frac{n^2}{3^n} \right) \). To use the ratio test we need to find \( a_n \) and \( a_{n+1} \). \textbf{What are these equal to?}

\[
a_n = \frac{n^2}{3^n} \quad \text{and} \quad a_{n+1} = \frac{(n+1)^2}{3^{n+1}}
\]

Putting these into \( L = \lim_{n \to \infty} \left( \frac{a_{n+1}}{a_n} \right) \) gives
\[ L = \lim_{n \to \infty} \left( \frac{a_{n+1}}{a_n} \right) \]

\[ = \lim_{n \to \infty} \left[ \frac{(n+1)^2}{3^{n+1}} \cdot \frac{n^2}{3^n} \right] \]

\[ = \lim_{n \to \infty} \left[ \frac{(n+1)^2}{3^{n+1}} \cdot \frac{3^n}{n^2} \right] \quad \text{[Turning the second fraction upside down]} \]

\[ = \lim_{n \to \infty} \left[ \frac{1}{3} \left( \frac{n+1}{n} \right)^2 \right] = \lim_{n \to \infty} \left[ \frac{1}{3} \left( 1 + \frac{1}{n} \right)^2 \right] = \frac{1}{3} \]

Since \( L = 1/3 \) which is less than 1 so the series converges.

(g) We need to test the series \( \sum \left( \frac{10^n}{n!} \right) \) for convergence. How?

By using the ratio test. Let \( a_n = \frac{10^n}{n!} \) then \( a_{n+1} = \frac{10^{n+1}}{(n+1)!} \). We have

\[ L = \lim_{n \to \infty} \left( \frac{a_{n+1}}{a_n} \right) \]

\[ = \lim_{n \to \infty} \left[ \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n} \right] \]

\[ = \lim_{n \to \infty} \left[ \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n} \right] = \lim_{n \to \infty} \left[ \frac{10}{n+1} \right] = 0 \]

Since \( L \) is less than 1 so the given series converges.

(h) Similarly for \( \sum \left( \frac{3^n}{(n+1)^2} \right) \) we use the ratio test. In this case \( a_n = \frac{3^n}{(n+1)^2} \) and

\[ a_{n+1} = \frac{3^{n+1}(n+1)}{(n+2)^2} \]. Putting these into \( L = \lim_{n \to \infty} \left( \frac{a_{n+1}}{a_n} \right) \) gives
We have \( L = 3 \) therefore the series diverges.

(i) We are given \( \sum \frac{n!}{(2n+1)!} \). We have \( a_n = \frac{n!}{(2n+1)!} \) therefore

\[
a_{n+1} = \frac{(n+1)!}{(2n+1)!} = \frac{(n+1)!}{(2n+1)!}
\]

Evaluating \( L \) we have

\[
L = \lim_{n \to \infty} \left( \frac{a_{n+1}}{a_n} \right)
\]

\[
= \lim_{n \to \infty} \left( \frac{(n+1)!}{(2n+3)!} \frac{n!}{(2n+1)!} \right)
\]

\[
= \lim_{n \to \infty} \left( \frac{(n+1)!}{(2n+3)!} \frac{(2n+1)!}{n!} \right)
\]

\[
= \lim_{n \to \infty} \left( \frac{n+1}{(2n+3)(2n+2)} \right)
\]

\[
= \lim_{n \to \infty} \left( \frac{n+1}{4n^2 + 10n + 6} \right)
\]

Dividing numerator and denominator by \( n \)

\[
\lim_{n \to \infty} \left( \frac{1 + 1/n}{4n + 10 + 6/n} \right) = 0
\]

Since \( L \) is equal to zero so the series converges.
(j) We need to test \( \sum \left( \frac{11^n}{2^{n+1} n} \right) \) for convergence. Let \( a_n = \frac{11^n}{2^{n+1} n} \) then \( a_{n+1} = \frac{11^{n+1}}{2^{n+2} (n+1)} \):

\[
L = \lim_{n \to \infty} \left( \frac{a_{n+1}}{a_n} \right) = \lim_{n \to \infty} \left( \frac{11^{n+1}}{2^{n+2} (n+1)} \times \frac{2^{n+1} n}{11^n} \right) = \lim_{n \to \infty} \left( \frac{11}{2 \left( 1 + \frac{1}{n} \right)} \right) = \frac{11}{2}
\]

We have \( L = \frac{11}{2} > 1 \) therefore the given series diverges.

12. In each case we show that \( L = \lim_{n \to \infty} \left( \frac{a_{n+1}}{a_n} \right) \) is equal to 1.

(a) We are given the series \( \sum \left( \frac{1}{n^3} \right) \) which means that \( a_n = \frac{1}{n^3} \) and \( a_{n+1} = \frac{1}{(n+1)^3} \):

\[
L = \lim_{n \to \infty} \left( \frac{a_{n+1}}{a_n} \right) = \lim_{n \to \infty} \left( \frac{1}{(n+1)^3} \times n^3 \right) = \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^3 = \left( \frac{1}{1 + \frac{1}{n}} \right)^3 = \frac{1}{1 + 0} = 1
\]

Hence the ratio test fails for this series.

(b) Similarly for \( \sum \left( \frac{1}{n+10} \right) \) we find that \( L = 1 \).

(c) For the given series \( \sum \left( \frac{1}{n^2 + 1} \right) \) we have \( a_n = \frac{1}{n^2 + 1} \) therefore \( a_{n+1} = \frac{1}{(n+1)^2 + 1} \):
\[ L = \lim_{n \to \infty} \left( \frac{a_{n+1}}{a_n} \right) \]

\[ = \lim_{n \to \infty} \left( \frac{1}{(n+1)^2 + 1} \times \frac{1}{n^2 + 1} \right) \]

\[ = \lim_{n \to \infty} \left( \frac{1}{(n+1)^2 + 1} \times \frac{n^2 + 1}{1} \right) \]

\[ = \lim_{n \to \infty} \left( \frac{n^2 + 1}{n^2 + 2n + 1} \right) \]

\[ = \lim_{n \to \infty} \left( \frac{1 + 1/n^2}{1 + 2/n + 2/n^2} \right) = \frac{1 + 0}{1 + 0 + 0} = 1 \]

Since \( L = 1 \) the ratio test fails.

13. (a) We are given \( \sum \left( \frac{2^n n!}{n^n} \right) \). Let \( a_n = \frac{2^n n!}{n^n} \) therefore \( a_{n+1} = \frac{2^{n+1} (n+1)!}{(n+1)^{n+1}} \). We have

\[ L = \lim_{n \to \infty} \left( \frac{a_{n+1}}{a_n} \right) \]

\[ = \lim_{n \to \infty} \left( \frac{2^{n+1} (n+1)!}{(n+1)^{n+1}} \div \frac{2^n n!}{n^n} \right) \]

\[ = \lim_{n \to \infty} \left( \frac{2^{n+1} (n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{2^n n!} \right) \]

\[ = \lim_{n \to \infty} \left( \frac{2^{n+1} (n+1)!}{(n+1)^{n+1}} \times \frac{n}{n+1} \right) \]

\[ = \lim_{n \to \infty} \left( 2 \left( \frac{n}{n+1} \right)^n \right) = 2 \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^n = \frac{2}{e} \]

Since \( L = \frac{2}{e} < 1 \) the given series converges.

ii Very similar to part i. We get \( L = \frac{3}{e} > 1 \) so the series diverges.

(b) We have \( \sum \left( \frac{x^n n!}{n^n} \right) \). Let \( a_n = \frac{x^n n!}{n^n} \) therefore \( a_{n+1} = \frac{x^{n+1} (n+1)!}{(n+1)^{n+1}} \). Determining \( L: \)
\[
L = \lim_{n \to \infty} \left( \frac{a_{n+1}}{a_n} \right) \\
= \lim_{n \to \infty} \left( \frac{x^{n+1}(n+1)!}{(n+1)^{n+1}} \div \frac{x^n n!}{n^n} \right) \\
= \lim_{n \to \infty} \left( \frac{x^{n+1}(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{x^n n!} \right) \\
= \lim_{n \to \infty} \left( x \left(1 + \frac{n}{n+1}\right)^n \right) \\
= x \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^n = \frac{x}{e}
\]

Remember the series converges if \( L \) is less than 1. We have

(i) \( 0 < x < e \)  
(ii) \( x > e \)