Chapter 3

Risk aversion

3.1 The Marschak-Machina triangle and risk aversion

One of the earliest, and most useful, graphical tools used to analyse choice under uncertainty was a triangular graph that was proposed by Jacob Marschak in 1950. The graph was later re-used extensively by Mark Machina during the 1980s to understand the results of experiments designed to find out whether real-life decisions can be explained by expected utility theory or not. Since this is a graphical exercise, it is necessary to study a reduced set of lotteries only. Concretely, we need to reduce the number of prizes down to $n = 3$, in order that the analysis can be carried out in an entirely two-dimensional environment. The assumption of only 3 prizes in any lottery is the greatest dimensionality that can be studied in a two-dimensional graph. If we place probabilities on the axes, lotteries can be represented by only two probabilities, since the third is the difference between the sum of the other two and the number 1. Concretely, it is convenient to eliminate the probability of the intermediate prize, writing it as $p_2 = 1 - p_1 - p_3$, thereby maintaining on the axes of the graph the probabilities of the two extreme prizes.

It is also necessary at this point to limit our lotteries to prizes over different quantities of a single good. Strictly speaking, this was not required in the previous chapter, so expected utility is valid for a wider range of options, but it is useful from here on. Given this, we shall simply assume that the only good in the model is money itself, and so all lotteries allocate prizes of different amounts of money. We shall use the generic variable w to represent such monetary amounts (and $\widetilde{w}$ to represent the corresponding random variable). The implication is that the utility function for prizes, $u(w)$, is just the indirect utility function from neoclassical demand theory (see Appendix B if you are unsure what indirect utility is or what properties it has). Finally, since we are assuming monetary prizes, the assumption of $w_i \succ w_j$ for $i < j$, can be expressed using normal inequalities as $w_1 > w_2 > w_3$.

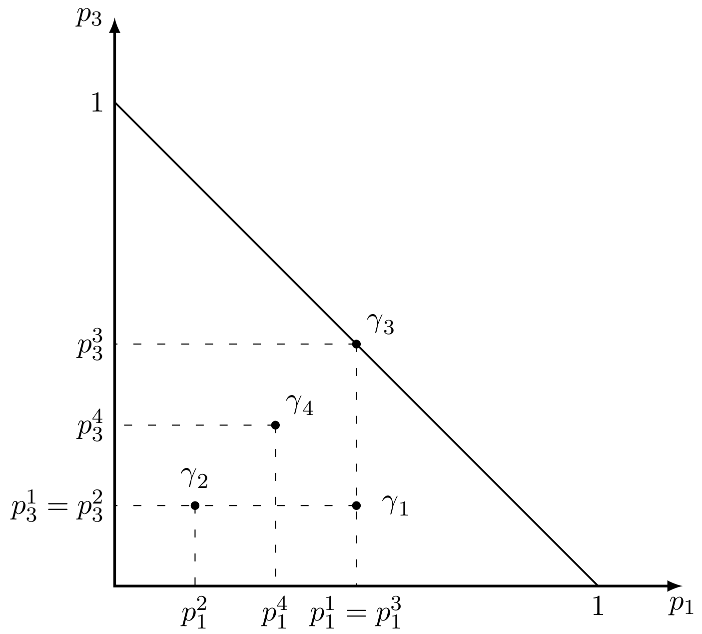

Figure 3.1 – A Marschak-Machina triangle

Recall that, at least throughout this chapter, the three numbers w_i $i = 1, 2, 3$ are fixed parameters at all times, and that different lotteries are represented by different probability vectors, $p^i \neq p^j$. Therefore, with $n = 3$, which as we have just mentioned allows us to write $p_2 = 1 - p_1 - p_3$, we can represent any given lottery as a point in the graphical space (p_1, p_3). This is done in Figure 3.1. Any lottery that lies on the horizontal axis has $p_3 = 0$, so the only possible prizes are w_1 and w_2. Similarly, any lottery that lies on the vertical axis indicates that the only possible prizes are w_2 and w_3. Finally, any lottery located on the hypotenuse of the triangle, where $p_1 + p_3 = 1$, and so $p_2 = 0$, implies that the only possible prizes are w_1 and w_3.

Thus, only when the lottery is located at a strictly interior point in the triangle are all three prizes possible (as is, for example, the case with lottery γ_1 in Figure 3.1).

- **Exercise 3.1**. Indicate as a distance in a Marschak-Machina triangle the probability p_2 for a strictly interior lottery.

- **Answer**. Draw a Marschak-Machina triangle, and place a dot in its interior somewhere. Label the coordinates of your dot, as read from the axes, as (p_1, p_3). But if we draw the line from your dot horizontally across until it touches the hypotenuse of the triangle, and then look at the value of the p_1-axis at that point, it must be the number $1-p_3$. This is just because the hypotenuse of the triangle defines the points such that $p_1 + p_3 = 1$, or $p_1 = 1-p_3$. Now, you have two points indicated on the horizontal axis – the point directly below your dot, which is the point p_1, and the point just located as $1 - p_3$. Along the horizontal axis, the distance from the origin to the point directly below your dot is the value of p_1, and the distance from the point just located as $1-p_3$ to the number 1 on the axis is the measure of p_3. And since the three probabilities must sum to 1, the value of p_2 is just the distance between the two points on your horizontal axis. If you like, from your dot, move directly to the right until you reach the hypotenuse. The distance travelled is p_2. Alternatively, from your dot move directly upwards until you reach the hypotenuse. Again, the distance travelled is p_2.

In order to understand the direction of preferences in the triangle, we need to use first-order stochastic dominance. Consider the two lotteries γ_2 and γ_1 in Figure 3.1. Since $p_3^1 = p_3^2$, it must be true that $p_1^1 + p_2^1 = p_1^2 + p_2^2$. But since $p_1^1 > p_1^2$, it turns out that lottery γ_1 first-order stochastically dominates γ_2, and so $\gamma_1 \succ \gamma_2$. Now consider γ_1 and γ_3. Since $p_3^1 < p_3^3$, we have $p_1^1 + p_2^1 > p_1^3 + p_2^3$, and again since $p_1^1 = p_1^3$, lottery γ_1 first-order stochastically dominates γ_3, and so it follows that $\gamma_1 \succ \gamma_3$. Finally, consider the lottery γ_4. Since $p_3^1 < p_3^4$ we have $p_1^1 + p_2^1 > p_1^4 + p_2^4$. But we also have $p_1^1 > p_1^4$, and so by first-order stochastic dominance, $\gamma_1 \succ \gamma_4$.

In short, first-order stochastic dominance indicates that more preferred lotteries in the triangle lie to the south-east. Of course, this also indicates that if we have two lotteries in the triangle that are indifferent to each other, then a straight line joining them must have strictly positive slope, and so in the triangle preferences can be represented by *indifference curves that have strictly positive slope.*

- **Exercise 3.2**. Consider a lottery that pays \$1 with probability $(1-p)$ and \$0 with probability p. Assume that a bettor is offered either one or two independent trials of this lottery. Call a single trial of the lottery L_1 and two independent trials L_2. Locate both L_1 and L_2 in a single Marschak-Machina triangle. Can you determine which of the two options is the most preferred for a risk averse bettor?

- **Answer**. The Marschak-Machina triangle would have the best prize equal to \$2, the intermediate prize equal to \$1 and the worst prize equal to \$0. Since L_1 offers a 0 probability of the best prize, it is located upon the vertical axis of the triangle, at a height of p. Lottery L_2 offers a probability of $(1-p)^2$ of the best prize, and a probability of p^2 of the worst prize. Thus L_2 locates at the strictly interior point defined by $p_1 = (1-p)^2$ and $p_3 = p^2$. Since $p^2 < p$, L_2 is located below and right of L_1. Under first-order stochastic dominance, L_2 is preferred to L_1.

Indeed, it is easy to get the exact equation for the slope of an indifference curve in the triangle, at least under expected utility. An indifference curve is defined as the set of points (p_1, p_3) such that $Eu(\widetilde{w}) = p_1u(w_1) + (1-p_1-p_3)u(w_2) + p_3u(w_3) = C$, where C is a constant and E is the expectations operator. Then, from the implicit function theorem, we have

$$\left.\frac{dp_3}{dp_1}\right|_{dEu(\widetilde{w})=0} = \frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)} > 0$$

Note that, since w_i $i = 1,2,3$ are constants, $u(w_i)$ $i = 1,2,3$ are also constants, and so the slope of an indifference curve is a positive constant (independent of the particular point (p_1, p_3) chosen). In other words, indifference curves in the Marschak-Machina triangle are straight lines, with higher valued curves lying to the south-east.

It is also interesting to compare indifference curves with the curves along which expected value is constant, $E\widetilde{w} = p_1w_1 + (1-p_1-p_2)w_2 + p_3w_3 = V$. In exactly the same way as above, we get

$$\left.\frac{dp_3}{dp_1}\right|_{dE\widetilde{w}=0} = \frac{w_1-w_2}{w_2-w_3} > 0$$

That is, the curves that maintain expected value constant (from now on, iso-expected value curves), are also straight lines with positive slope. The interesting question is, how do the indifference curves and the iso-expected value curves compare to each other? The answer depends entirely upon the concavity of the utility function, $u(w)$. Let's see how.

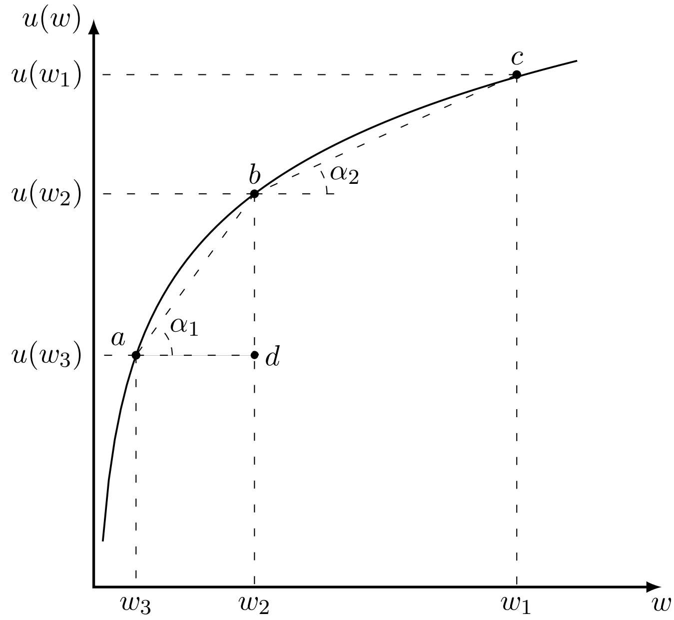

Figure 3.2 – Concave utility function

Figure 3.2 shows a typical concave utility function, along with the three levels of wealth $w_1 > w_2 > w_3$. If we draw the line segments joining point a to point b, and point b to point c, then due to the concavity of the utility function, the slope of the line joining a to b must be greater than the slope of the line joining b to c. That is, $\alpha_1 > \alpha_2$. We can measure these two slopes using some simple geometry. Consider the triangle formed by the three points a, b and d. The slope of the line (actually, the tangent of the angle at α_1) joining a to b is equal to the length of the opposite side (the distance from d to b) divided by the length of the adjacent side (the distance from a to d). But these two distances are, respectively, $u(w_2) - u(w_3)$ and $w_2 - w_3$. Thus, we have $\alpha_1 = \frac{u(w_2)-u(w_3)}{w_2-w_3}$. In exactly the same way, we have $\alpha_2 = \frac{u(w_1)-u(w_2)}{w_1-w_2}$. And since $\alpha_1 > \alpha_2$, we get $\frac{u(w_2)-u(w_3)}{w_2-w_3} > \frac{u(w_1)-u(w_2)}{w_1-w_2}$, which rearranges directly to $\frac{w_1-w_2}{w_2-w_3} > \frac{u(w_1)-u(w_2)}{u(w_2)-u(w_3)}$.

In words, if $u(w)$ is strictly concave, then the iso-expected value lines are steeper than the indifference curves. Such a situation is drawn in Figure 3.3. Clearly, if the utility function is convex rather than concave, then the iso-expected value lines would work out to be less steep than the indifference curves, and if the utility function were linear, then the two sets of curves in the Marschak-Machina graph would coincide exactly.

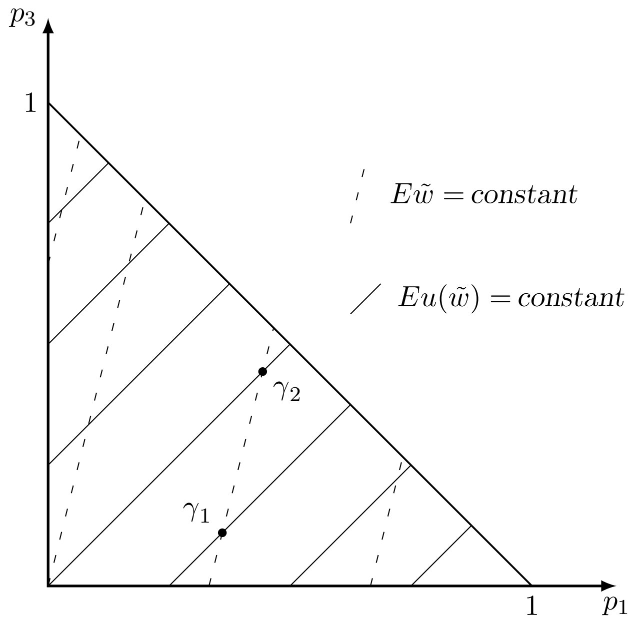

Figure 3.3 – Expected value and expected utility in the Marschak-Machina triangle under concave utility

Since the most reasonable assumption on the utility function is that it is concave (decreasing marginal utility of wealth), from now on we shall assume that this is so, and so we shall be dealing exclusively with situations like that of Figure 3.3.

Now, consider two lotteries with the same expected value, say γ_1 and γ_2 in Figure 3.3. Since $u(w)$ is concave, we get $\gamma_1 \succ \gamma_2$, that is $U(\gamma_1) > U(\gamma_2)$. Apart from the difference in expected utility, the two lotteries also differ as far as their statistical variance is concerned.

Variance is defined as $var(\gamma) = \sigma^2(\gamma) = \sum p_i(w_i - E\widetilde{w})^2$. In fact, it turns out that $\sigma^2(\gamma_2) > \sigma^2(\gamma_1)$. To see why, it is necessary to consider the derivative of $\sigma^2(\gamma)$ with respect to p_1 conditional upon $E\widetilde{w}$ remaining constant (this is suggested as problem 1 at the end of the chapter). Never-the-less, note that as we increase p_1 along a particular iso-expected value line, we need to increase p_3 and therefore decrease p_2. This corresponds to a displacement of probability weight from the centre of the distribution to the extremes, which implies an increase in variance.

In short, we have reached the following important conclusion: *if the utility function is strictly concave, then an increase in variance while holding the expected value constant implies a decrease in expected utility.*[1] Economists say that such preferences display *risk aversion*, since it is normal to associate variance with risk. Thus, concavity of the utility function is equivalent to risk aversion. Of course, if $u(w)$ were linear, then we would have a risk neutral preference, and if it were convex we have a preference for risk (sometimes referred to as risk loving).

- **Exercise 3.3**. Draw in a Marschak-Machina triangle the lotteries corresponding to the Allais paradox, and show how the paradox cannot be consistent with expected utility.

- **Answer**. If we set $w_1 = 25,\ w_2 = 5$ and $w_3 = 0$, the four lotteries of the Allais paradox can be easily located in the Marschak-Machina triangle. This has been done in Figure 3.4. What we should note is that a straight line that connects the two lotteries in the first choice (γ_A and γ_B) will have exactly the same slope as a straight line that connects the two lotteries in the second choice (γ_C and γ_D). Concretely, the slope of the two connecting lines is 0.1. However, if the individual making the choices is an expected utility maximiser, then we know that his indifference curves over the entire probability space are straight lines with common slope, and so if $\gamma_A \succ \gamma_B$, then these indifference curves must have a slope that is less than 0.1. But then, we would have to conclude that $\gamma_C \succ \gamma_D$. One possible way to explain the apparent paradox is for the individual to have a preference over lotteries that corresponds to indifference curves that "fan

[1]Although it may not be so obvious, it is also true that a decrease in expected value while maintaining variance constant will reduce expected utility. Proving this in the triangle is not easy, since the iso-variance curves are conics.

out", in the sense that they are steeper and steeper as we move upwards and to the west in the triangle. Such preferences cannot correspond to expected utility.

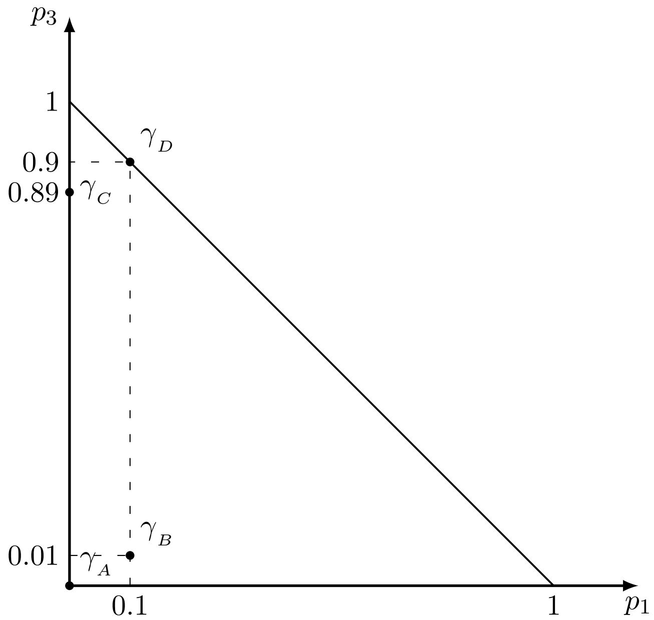

Figure 3.4 – Allais paradox in the Marschak-Machina triangle

3.2 The contingent claims environment

In short, expected utility theory asserts that, so long as the individual's preferences satisfy a short list of very reasonable assumptions, then the utility that should be attached to a lottery is nothing more than the mathematical expectation of the utility of each prize. If we restrict our attention (as we will from now on) to lotteries with only two prizes, each of which is an amount of wealth, say prize w_1 with probability $1-p$ and w_2 with probability p, then the utility of this situation is

$$Eu(\widetilde{w}) = (1-p)u(w_1) + pu(w_2) \tag{3.1}$$

Of course, the utility function in question would really be the indirect utility function, and so $u(w)$ denotes the "utility of wealth", or the "utility of money".

This is a simple case of a utility function that is separable, and this makes it easy to study in the same type of graphical environment that is typically used in undergraduate consumer theory under certainty. The graph in question is often called the “contingent claims” graph, which assumes probabilities to be fixed and prizes to be variable monetary amounts.

The first important analysis based on variable prizes and fixed probabilities is the model of Nobel Laureates Ken Arrow and Gerard Debreu, where general equilibrium is extended to account for uncertainty. In that model, different “states of nature” are thought of as different markets for contingencies. The model extends the space of goods by understanding that there is no formal difference between two different goods, and the same good at two different locations, or at two different states of nature. Thus, uncertainty is just an increase in the number of different goods present in a model. The Arrow-Debreu model is known as the “contingent claims” environment.[2]

The fundamental idea is simple, even more so in two-dimensional space. We begin by establishing a set of possible states of nature, where a state of nature is simply a full and complete description of all relevant aspects contingent upon a given outcome of a stochastic process. We also need to establish the probability density over the possible states of nature. For example, an investor in the stock exchange knows that the price of his shares may go up (state 1) or go down (state 2).[3] As soon as we establish the probability that the shares will increase in price, then we have a properly defined contingent claims environment. For the type of problem that we will be interested in here, we shall simply consider an individual’s wealth, w. We shall assume that there are only two possible states of nature, state 1 and state 2, and that in state i the individual’s wealth is w_i $i = 1, 2$. We shall denote the probability of state 2 as p, and thus the probability of state 1 is $1 - p$. The relevant utility function for this type of problem is the individual’s indirect utility, which we shall denote by $u(w)$.

[2]The contingent claims model is useful when choices can lead to alterations in the set of prizes. For example, take the case of a person who faces an initial lottery in which he can lose x with a given probability p, and lose nothing with probability $1 - p$. If he insures half of this loss at a premium of q, then he now gets a loss of $0.5x + q$ with probability p and a loss of q with probability $1 - p$. Same probabilities, different prizes.

[3]Of course, with this example it is possible to define a much richer set of states of nature – the price of one share goes up by 1% and that of another goes down by 2%, and so on.

Continuing from what we have already done in the earlier sections of this book, we shall assume always (unless otherwise stated) that this function is strictly increasing and concave, $u'(w) > 0$ and $u''(w) < 0$.

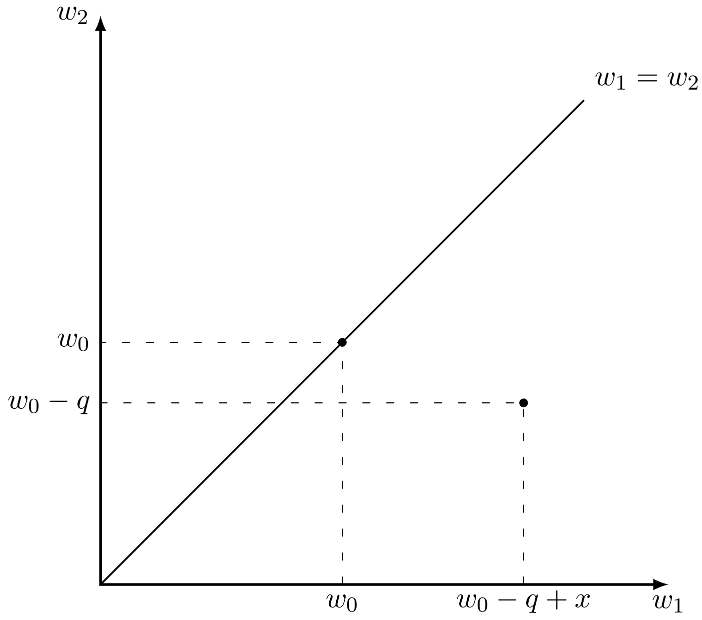

Figure 3.5 – Contingent claims space

In a two-dimensional graph, we can represent the individual's wealth in each state of nature on the axes (see Figure 3.5). The initial endowment of the individual is often referred to as the initial risk allocation. It is customary that, when initially $w_1 \neq w_2$, then we define the state of nature with less wealth to be state 2, that is, we would define our states such that $w_2 < w_1$. The diagonal line passing through the origin of the graph (the line with slope equal to 1) is known as the "certainty line", since along it are all the vectors of wealth such that $w_1 = w_2$. Clearly, independently of the probabilities of receiving the two different wealth levels, if they are both equal then final wealth is known with probability 1 (i.e., with certainty) as $w = w_1 = w_2$.

As an example, in Figure 3.5 two situations are shown. On the one hand, we show the case of an individual with certain wealth of w_0, and on the other hand the situation of an individual with a certain wealth of w_0 plus a lottery ticket that costs q to purchase and that pays a prize of $x > 0$ with probability $1 - p$ and a prize of 0 with probability p. We assume that $q < x$ so that the lottery ticket is a logically valid option to consider. The wealth vector contingent upon the outcome of the lottery is $(w_1, w_2) = (w_0 - q + x, w_0 - q)$. Since $q < x$, even though the risk distribution before purchasing the lottery is on the certainty line, the distribution achieved after purchasing it lies beneath the certainty line. The important point to note about the contingent claims environment is simply that the individual will only actually receive the wealth indicated on one of the axes, that is, the wealth level of only one of the components of the vector w, rather than both components as is the case in traditional two-dimensional consumer theory under certainty.

To begin with, let us reconsider the expected value and variance of any particular point in contingent claims space. By definition, where E represents the expectations operator, the expected value of a point w is

$$E\widetilde{w} = pw_2 + (1-p)w_1 \tag{3.2}$$

Clearly, this equation presents a structure that is identical to a budget constraint in traditional consumer theory, but where now instead of prices we have probabilities. Using this comparison (or, if you like, just use the implicit function theorem) it is immediate that, in the contingent claims space, the slope of a line that holds expected value constant is just

$$\left.\frac{dw_2}{dw_1}\right|_{dE\widetilde{w}=0} = -\frac{(1-p)}{p} < 0$$

So an iso-expected value line in this space is a straight line with negative slope. Any particular one of them (different lines corresponding to different levels of expected value) will cut through the certainty line at only one point. Let us identify this point as $w_1 = w_2 = \overline{w}$, and then we have $E\widetilde{w} = p\overline{w} + (1-p)\overline{w} = \overline{w}$. This tells us that the further an iso-expected value line lies from the origin of the graph, the greater is the expected value it represents.

Second, the variance of a point w is defined by

$$\sigma^2(\widetilde{w}) = p(w_2 - E\widetilde{w})^2 + (1-p)(w_1 - E\widetilde{w})^2$$

Using (3.2), the variance $\sigma^2(\widetilde{w})$ is equal to

$$\begin{aligned} &p(w_2 - pw_2 - (1-p)w_1)^2 + (1-p)(w_1 - pw_2 - (1-p)w_1)^2 \\ &\quad = p((1-p)(w_2 - w_1))^2 + (1-p)(p(w_1 - w_2))^2 \\ &\quad = p(1-p)^2(w_2 - w_1)^2 + (1-p)p^2(w_1 - w_2)^2 \end{aligned}$$

And since $(w_2 - w_1)^2 = (w_1 - w_2)^2$, it turns out that

$$\begin{aligned} \sigma^2(\widetilde{w}) &= p(1-p)\left[(1-p)(w_1 - w_2)^2 + p(w_1 - w_2)^2\right] \\ &= p(1-p)(w_1 - w_2)^2 \end{aligned}$$

Therefore, the iso-variance curves in contingent claims space are also straight lines, but with slope equal to 1, since directly from the implicit function theorem we can calculate that

$$\left.\frac{dw_2}{dw_1}\right|_{d\sigma^2=0} = -\frac{2p(1-p)(w_1 - w_2)}{-2p(1-p)(w_1 - w_2)} = 1$$

The certainty line is a particular example of an iso-variance line. It is the line along which variance is equal to 0. Iso-variance lines that lie further away from the certainty line, in either direction, indicate a greater variance since they correspond to a greater value of $w_1 - w_2$, as is shown in Figure 3.6.

Now we can consider preferences. As we have already argued, preferences in the model are given by expected utility. Therefore, the utility of a vector w is $Eu(\widetilde{w}) = pu(w_2) + (1-p)u(w_1)$. An indifference curve maintains expected utility constant, $dEu(\widetilde{w}) = 0$, and so again we only need to apply the implicit function theorem to see that

$$\left.\frac{dw_2}{dw_1}\right|_{dEu(\widetilde{w})=0} = -\frac{(1-p)u'(w_1)}{pu'(w_2)} \equiv MRS(w) \tag{3.3}$$

Here, $MRS(w)$ indicates the marginal rate of substitution at the point w.

Since we are assuming that the utility function is increasing and concave, it is not difficult to prove that expected utility is concave in the vector (w_1, w_2) (see problem 5 at the end of the chapter). So, for any two points in the contingent claims space, w^1 and w^2, and for any λ that satisfies $0 < \lambda < 1$, it is true that

$$Eu(\lambda\widetilde{w}^1 + (1-\lambda)\widetilde{w}^2) > \lambda Eu(\widetilde{w}^1) + (1-\lambda)Eu(\widetilde{w}^2)$$

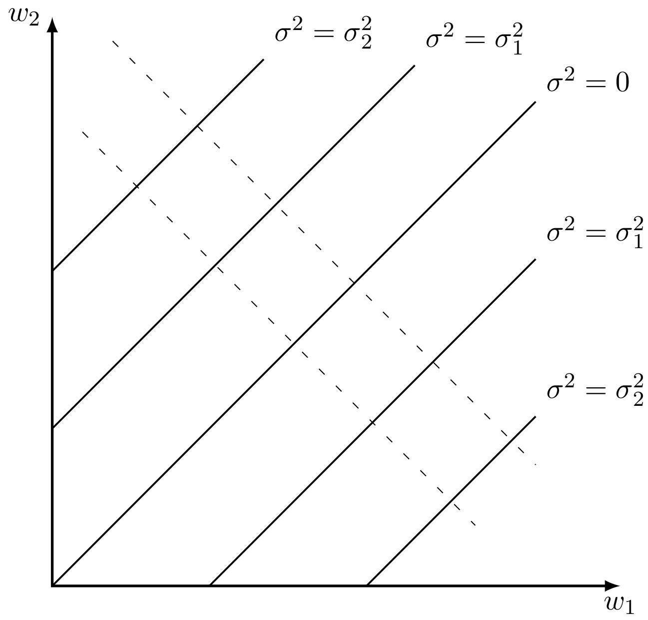

Figure 3.6 – Expected value and variance lines in the contingent claims graph

Given that, we know that the corresponding indifference curves are strictly convex.

Finally, note that since the indifference curves have negative slope, each one must cut the certainty line at exactly one point. If we denote this point by $w_1 = w_2 = \widehat{w}$, then we have

$$Eu(\widetilde{w}) = pu(\widehat{w}) + (1-p)u(\widehat{w}) = u(\widehat{w})$$

Since we have assumed $u'(w) > 0$, it is now evident that indifference curves that are further from the origin indicate more preferred vectors, since they are consistent with a greater level of expected utility.

If we draw some indifference curves corresponding to a strictly concave utility function together with iso-expected value and iso-variance lines, then it becomes immediate that the individual displays what is known as *risk aversion* (see Figure 3.7). Risk is taken as being analogous to variance, and so risk aversion is the characteristic that leads individuals to dislike variance at any given expected value. First, note that from the equation for the marginal rate of substitution (3.3), the slope of an indifference curve at the point at which it cuts

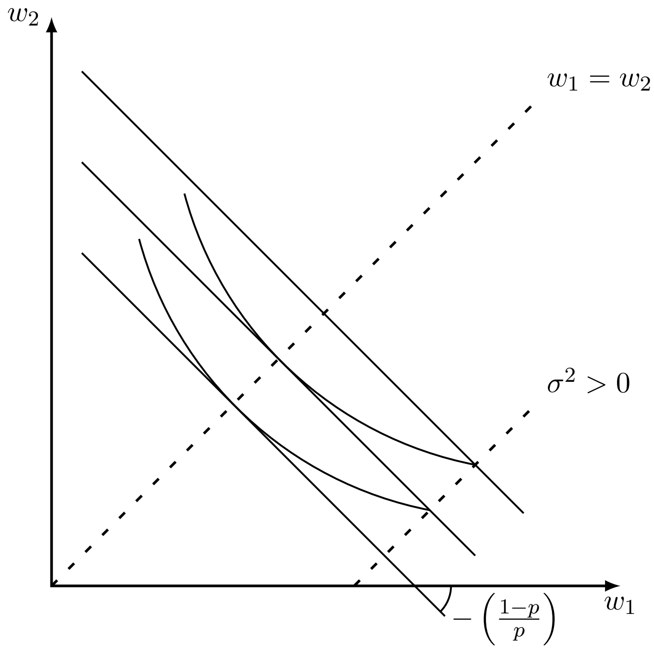

Figure 3.7 – Expected utility indifference curves with risk averse preferences

the certainty line is equal to $-\frac{(1-p)}{p}$, which is the same slope as an iso-expected value line. Therefore, we can directly deduce that the unique solution to the problem of choosing freely from all lotteries with an expected value that is less than or equal to some particular amount, say $\overline{w}$, is the lottery that gives an expected value of exactly $\overline{w}$ but with zero variance – our decision maker is clearly showing a dislike for variance, that is, he is risk averse. In order to see this in another way, consider a movement along an iso-expected value line towards lotteries of ever greater variance (i.e., movements away from the certainty line). When the indifference curves are convex, each such successive movement implies moving to a lower indifference curve, which again directly implies risk aversion as defined above. In the contingent claims environment, it is also immediate to see that an increase in expected value that holds variance constant will always increase expected utility.

- **Exercise 3.4**. Consider your own preferences for simple lotteries. Imagine you are offered the choice between a coin-toss where the outcome is win a dollar on heads, lose a dollar on tails. Would you voluntarily accept this lottery? How about win two dollars on heads, lose a dollar on tails? Try to answer the following question honestly. You are offered to voluntarily play a lottery in which on heads you win x dollars, and on tails you lose one dollar. What is the smallest value of x for which you would play this lottery? What is the expected value of the lottery, and what is its variance? Think about what your answers imply for your own preference towards risk.

- **Answer**. This question relates to your own personal preferences, so there is no one correct answer here. Different people will answer the question differently. However, most people would not voluntarily accept the coin-toss lottery that pays one dollar on heads and that costs one dollar on tails. If that is true for you, then your preferences display risk aversion, at least for this small stakes range of wealth. If I am asked about a coin-toss lottery in which I lose a dollar on tails and gain x dollars on heads, I would set my minimum value of x at something around \$1.50. The expected value of the lottery is $0.5 \times x + 0.5 \times (-1) = 0.75 - 0.5 = 0.25$. The variance is $0.5 \times 0.5(1.5 + 1)^2 = \frac{2.5^2}{4} = 1.5625$. My preferences display risk aversion since in order to be indifferent between playing (having a variance of 1.5625) and not playing (having a variance of 0), I require a strictly positive gain in expected value. Graphically, the minimally acceptable lottery is located below the certainty line, and above the expected value line of not playing. Since my indifference curve for not playing cuts the certainty line at the same place as the expected value line for not playing, it must be a convex curve in order to also go through the lottery point (recall, I am indifferent whether I play or not).

- **Exercise 3.5**. A classic problem in the economics of risk is the choice of the split of initial risk-free wealth between an asset with a risk-free return, and one with a risky return. Each dollar invested in the risk-free asset yields, say, $(1 + t)$ dollars for sure, while each dollar invested in the risky asset yields, say, $(1 + r)$ dollars with probability $(1 - p)$ and $(1 - r)$ dollars with probability p. Assume that r and t are both positive numbers, and that that the investor can split his money, putting some in the risk-free asset and the rest in the risky asset. Assume that the risky asset has a higher expected return than the risk-free asset. Can the risk-free asset ever dominate the risky one, in the sense that the investor would invest only in the risk-free asset and not in the risky asset at all?

- **Answer**. Whether or not the risky asset will be purchased at all depends upon the relationship between the expected value of the risky asset and that of the risk-free asset. The risky asset will only be included in the optimal portfolio if it has a strictly greater expected value than the risk-free asset. This happens if $(1-p)(1+r) + p(1-r) > 1+t$. When this happens, it is *always* optimal to include some of the risky asset in the optimal portfolio, regardless of risk aversion. It may also happen that the risky asset is the only one in the optimal portfolio (a corner solution). The problem is relatively simple to solve graphically (see Figure 3.8).

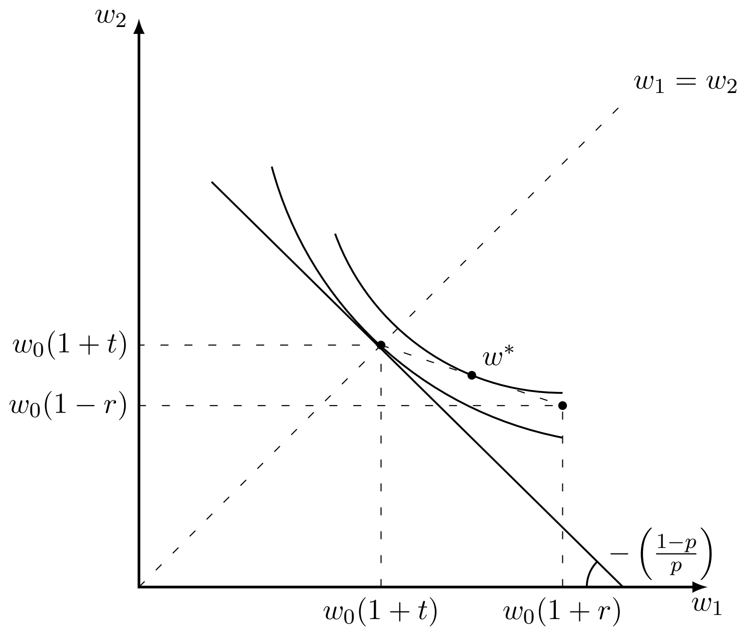

Figure 3.8 – Optimal choice between a risky and a risk-free asset

Since the expected return on the risky asset is greater than the expected return on the risk-free asset, we know that the point corresponding to all wealth invested in the risky asset, which lies below the certainty line, must lie above the expected value line passing through the point on the certainty line corresponding to all wealth invested in the risk-free asset. Thus, the straight line joining these two points (the line showing all possible investment opportunities as wealth is spread over the two investments) is less steep (flatter) than the expected value line of the risk-free investment. But since the slope of the risk-free expected value line is simply the ratio of state contingent probabilities, we also know that the indifference curve at the risk-free investment is steeper than the market opportunities line. Thus the tangency between the market opportunities line and the indifference curve must occur below the certainty line, that is, some money is always invested in the risky asset. Curiously, the result that some risk will always be purchased is independent of exactly how risk averse the individual is, and how slight might be the expected value advantage of the risky asset.

3.3 Measures of risk aversion

Now that the concept of risk aversion has been formally introduced, it makes sense to analyse it in greater detail. One of the most interesting question about risk aversion is whether or not we can characterise different individuals according to their risk aversion. That is, if any two individuals can be ranked, or ordered, according to who is more risk averse. In order to do this, consider Figure 3.9, in which we represent the situation of an individual with an initial situation of pure risk-free wealth of w_0. The initial wealth endowment is given by the point $w^0 = (w_1^0, w_2^0) = (w_0, w_0)$. The indifference curve that passes through this point cuts the contingent claims space into two separate parts; the points that lie below the endowed indifference curve (lotteries that are less preferred to the endowment point w^0, i.e. $w : w^0 \succ w$) and points that are on or above the endowed indifference curve (lotteries that are at least as preferred as w^0, i.e., $w : w \succsim w^0$). We shall refer to the set $A(w^0) = \{w : w \succsim w^0\}$ as the *acceptance set*, since it indicates all the lotteries that the individual would accept, voluntarily, in exchange for his endowment.

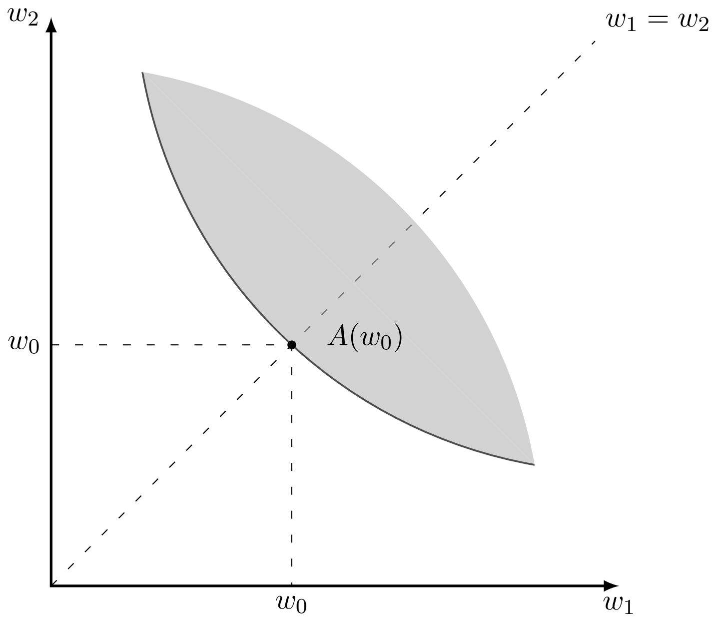

Figure 3.9 – An acceptance set

Absolute risk aversion

Now, consider two individuals who are identical in all but their utility function. In particular, the two individuals share the same endowment point, and the same probabilities of the two states of nature. We have just seen that independently of the particular utility function, the slope of an indifference curve as it crosses the certainty line is always equal to $-\frac{(1-p)}{p}$, and so the frontiers of the two acceptance sets are necessarily tangent to each other at the common endowment point w^0. Assume now that one of the acceptance sets is a sub-set of the other, say $A_i(w^0) \subset A_j(w^0)$, then all of the lotteries that are acceptable to individual i are also acceptable to individual j, but the opposite is not true. There exist lotteries that are accepted by j but that are rejected by i in a proposed exchange for the endowment point w^0. Concretely, for any particular expected value the lotteries that are acceptable to j but that are rejected by i are those with the greatest variance within j's acceptance set. They are the lotteries that correspond to greater risk. In this case, it is natural to say that i is, locally (i.e., around w^0), more risk averse than j.

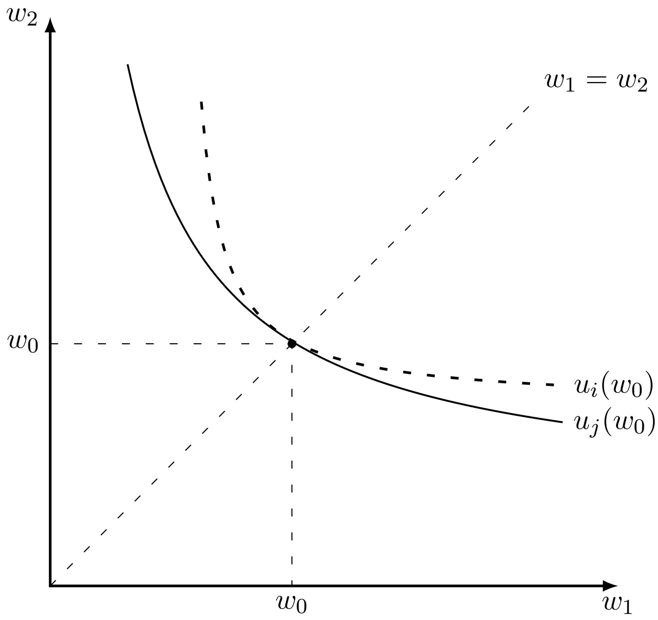

Figure 3.10 – Greater risk aversion

Graphically, if individual i is more risk averse than is individual j, then the indifference curve of i that passes through the endowment point will be, at least locally, more convex than the indifference curve of j passing through the same point (Figure 3.10). Let's formalise this idea.

To begin with, from equation (3.3), the first derivative of an indifference curve at any point is

$$\left.\frac{dw_2}{dw_1}\right|_{dEu(\widetilde{w})=0} = -\frac{(1-p)u'(w_1)}{pu'(w_2)}$$

Differentiating a second time with respect to w_1, we find

$$\left.\frac{d^2w_2}{d(w_1)^2}\right|_{dEu(\widetilde{w})=0} = -\left(\frac{1-p}{p}\right)\left(\frac{u''(w_1)u'(w_2) - u'(w_1)u''(w_2)\left(\frac{dw_2}{dw_1}\right)}{\left[u'(w_2)\right]^2}\right)$$

Substituting in the point $w_2 = w_1 = w_0$ yields

$$\left.\frac{d^2 w_2}{d(w_1)^2}\right|_{dEu(\widetilde{w})=0} = -\left(\frac{1-p}{p}\right)\left(\frac{u''(w_0)u'(w_0) - u'(w_0)u''(w_0)\left(-\frac{(1-p)}{p}\right)}{[u'(w_0)]^2}\right)$$

$$= -\left(\frac{1-p}{p}\right)\left(\frac{u''(w_0)u'(w_0)\left(1+\frac{(1-p)}{p}\right)}{u'(w_0)^2}\right)$$

$$= -\frac{u''(w_0)}{u'(w_0)}f(p)$$

$$\equiv R^a(w_0)f(p)$$

where $f(p) = \frac{1-p}{p^2}$. The important point to note is that, since the two individuals share the same probability p, if their endowed indifference curves have different second derivatives at the endowment point, then this difference is due entirely to the term $R^a(w_0)$. Given our assumptions that $u'(w) > 0$ and $u''(w) < 0$, it turns out that $R^a(w_0)$ is positive. $R^a(w_0)$ is known in the economics literature as the *Arrow-Pratt measure of absolute risk aversion*, and if $R^a_i(w_0) > R^a_j(w_0)$ then individual i is more risk averse than individual j.

We can use the measure of absolute risk aversion to point out an important aspect of expected utility that was not mentioned earlier. Clearly, if two utility functions, $u_i(w)$ and $u_j(w)$, are to represent the same preferences in an uncertain or risky environment, then they must share exactly the same set of indifference curves in contingent claims space. But this can happen only if both functions have the same measure of absolute risk aversion at any given wealth w, that is, we require $R^a_i(w) = R^a_j(w)$ for all w.

Now, traditional consumer theory under certainty teaches us that a utility function is an ordinal concept, that is, it is only useful for ordering consumption bundles from the least to the most preferred. If, in a certainty environment, we have $w_i \succ w_j$, then in principle we can use a utility function that returns $u(w_i) = 4$ and $u(w_j) = 2$ or another that returns $u(w_i) = 37$ and $u(w_j) = 9.6$. The only important point is that $u(w_i) > u(w_j)$, and not the difference between the two utility values, $u(w_i) - u(w_j)$. In a model of choice under certainty, we say that if a utility function $u(w)$ represents preferences $\succsim$ in the sense that $u(w_i) \geq u(w_j)$ if and only if $w_i \succsim w_j$, then any $f(u(w))$

with $f'(u) > 0$ will also represent the same preferences. A composite function of the form $z(w) \equiv f(u(w))$ with $f'(u) > 0$ is known as a *positive monotone transformation* of $u(w)$.

Let's now go back to our uncertain environment (just for now, let us consider an n state world, rather than a strictly 2 state world). If two utility functions for wealth $u_i(w)$ and $u_j(w)$ are to represent the same preferences over lotteries, then it must be true that the two functions always give the same ordering over lotteries, or in other words, that the two expected utilities are related by a positive monotone transformation:

$$\sum_{k=1}^{n} p_k u_i(w_k) = H\left(\sum_{k=1}^{n} p_k u_j(w_k)\right) \text{ with } H'(\cdot) > 0$$

Differentiating with respect to (any) w_k, we have

$$u_i'(w_k) = H'(\cdot)u_j'(w_k) \ \ \forall w_k$$

and so

$$H'(\cdot) = \frac{u_i'(w_k)}{u_j'(w_k)} \ \ \forall w_k \tag{3.4}$$

Differentiating (3.4) yields

$$H''(\cdot) = \frac{u_i''(w_k)u_j'(w_k) - u_i'(w_k)u_j''(w_k)}{\left[u_j'(w_k)\right]^2} \ \ \forall w_k$$

But recall that if the two functions are to represent the same preferences over lotteries, then it must hold that $R_i^a(w_k) = R_j^a(w_k)$ for all w_k, that is

$$-\frac{u_i''(w_k)}{u_i'(w_k)} = -\frac{u_j''(w_k)}{u_j'(w_k)} \ \ \Rightarrow \ \ u_i''(w_k)u_j'(w_k) = u_i'(w_k)u_j''(w_k) \ \ \forall w_k$$

and so clearly it would have to hold that

$$H''(\cdot) = 0 \ \ \forall w_k$$

In words, if the two utility functions for wealth are to represent the same preferences for lotteries, then we can admit only functions $H(\cdot)$ relating the implied expected utilities that are linear. Therefore

$$\sum_{k=1}^{n} p_k u_i(w_k) = H\left(\sum_{k=1}^{n} p_k u_j(w_k)\right) = a\sum_{k=1}^{n} p_k u_j(w_k) + b$$

where $a > 0$ from (3.4). Now, since $b = \sum_{k=1}^{n} p_k b$, we have

$$\sum_{k=1}^{n} p_k u_i(w_k) = a \sum_{k=1}^{n} p_k u_j(w_k) + \sum_{k=1}^{n} p_k b = \sum_{k=1}^{n} p_k \left(a u_j(w_k) + b\right)$$

that is

$$u_i(w) = a u_j(w) + b \quad \text{with} \quad a > 0$$

Again, in words, if $u_i(w)$ and $u_j(w)$ represent the same preferences in a problem of choice under risk or uncertainty, then they must be related linearly. This implies that the incorporation of the dimension of uncertainty to a choice model reduces the set of admissible utility functions by reducing the generality of the type of transformation that can be used. Instead of any positive monotone transformation, we are now restricted to those that are linear. This important difference between utility representations in problems of choice under certainty and under uncertainty has led to the uncertainty utility function becoming known as a "von Neumann-Morgenstern" utility function, named after the economists who first formally proved the validity of expected utility theory.

In short, if we assume two individuals i and j with different utility functions in the sense that there are no two numbers $a > 0$ and b such that $u_i(w) = a u_j(w) + b$, then this difference implies that $R_i^a(w) \neq R_j^a(w)$. In that case we can name our individuals such that $R_i^a(w) > R_j^a(w)$, that is, individual i is (locally in the neighbourhood of a level of wealth w) more risk averse than individual j.

- **Exercise 3.6**. Prove that if $f(u)$ is a strictly increasing and concave function ($f'(u) > 0$ and $f''(u) < 0$), then the utility function $v(w) \equiv f(u(w))$ is more risk averse than the utility function $u(w)$.

- **Answer**. The first derivative of $v(w) = f(u(w))$ with respect to w is $v'(w) = f'(u)u'(w)$. Differentiating again with respect to w yields $v''(w) = f''(u)u'(w)^2 + f'(u)u''(w)$. Thus, by construction the Arrow-Pratt measure of absolute risk aversion for utility function $v(w)$ is $R_v^a(w) = -\frac{f''(u)u'(w)^2 + f'(u)u''(w)}{f'(u)u'(w)}$ which simplifies to $-\frac{f''(u)u'(w)^2}{f'(u)u'(w)} - \frac{f'(u)u''(w)}{f'(u)u'(w)}$ or $R_f^a(u)u'(w) + R_u^a(w)$. Since $R_f^a(u)u'(w) > 0$ it turns out that $R_v^a(w) > R_u^a(w)$ for all w. So indeed $v(w)$ is more risk averse than $u(w)$.

Note that $R^a(w)$ is a properly defined function that returns a value for any given scalar[4] w, since we could have used any particular point on the certainty line as our endowment in the above argument. Shortly we shall discuss the derivatives of $R^a(w)$, which are of immense importance to the economics of risk and uncertainty.

Relative risk aversion

The word "absolute" in the name of $R^a(w)$ is due to the fact that the lotteries used to find it are absolute, that is, they are lotteries whose prizes w_1 and w_2 are absolute quantities of money. There exists a second type of lottery, denominated as relative lotteries, whose prizes are expressed in relative terms to the initial situation. For example, the lottery defined by $\eta_r = (r_1, r_2, 1-p, p)$ is a relative lottery if the prizes are $r_i w$ for $i = 1, 2$ and for any particular initial w.

In the space of the r_i we can represent the indifference curves for relative lotteries, and these curves are closely related to those of absolute lotteries. To see this, note that the expected utility of a relative lottery is

$$Eu(\widetilde{r}w) = pu(r_2 w) + (1-p)u(r_1 w)$$

By the implicit function theorem, we have

$$\left.\frac{dr_2}{dr_1}\right|_{dEu=0} = -\frac{(1-p)u'(r_1 w)w}{pu'(r_2 w)w} = -\frac{(1-p)u'(r_1 w)}{pu'(r_2 w)}$$

For any relative lottery that offers certainty (that is, $r_1 = r_2$), we get the result that the slope of the indifference curve is equal to $-\frac{(1-p)}{p}$, just as in the case of absolute lotteries. The second derivative of an indifference curve in the space of relative lotteries is

$$\left.\frac{d^2 r_2}{d(r_1)^2}\right|_{dEu=0} =$$
$$-\left(\frac{1-p}{p}\right)\left(\frac{wu''(r_1 w)u'(r_2 w) - u'(r_1 w)wu''(r_2 w)\left(\frac{dr_2}{dr_1}\right)}{(u'(r_2 w))^2}\right)$$

[4] Earlier we used w to indicate a wealth vector, and now it is being used to indicate a scalar. From the context of the analysis it should always be clear what the exact dimensionality of w is being assumed.

Now, at any lottery such that $r_1 = r_2 = r$, we get

$$\begin{aligned}
\left.\frac{d^2 r_2}{d(r_1)^2}\right|_{dEu=0} &= \\
&-\left(\frac{1-p}{p}\right)\left(\frac{wu''(rw)u'(rw) - u'(rw)wu''(rw)\left(-\frac{1-p}{p}\right)}{(u'(rw))^2}\right) \\
&= -\left(\frac{1-p}{p}\right)\left(\frac{wu''(rw)u'(rw)\left(1+\frac{1-p}{p}\right)}{(u'(rw))^2}\right) \\
&= -\left(\frac{1-p}{p}\right)\left(1+\frac{1-p}{p}\right)\left(\frac{wu''(rw)}{u'(rw)}\right) \\
&= -\frac{wu''(rw)}{u'(rw)} f(p) \\
&\equiv R^r(w) f(p)
\end{aligned}$$

Note that when $r = 1$, we have $R^r(w) = wR^a(w)$. But if we assume (as before) that our individual starts off with an initial wealth w that is risk-free, then the relevant point in the space of relative lotteries to represent such an endowment is exactly the certainty lottery with $r = 1$, and so this is the lottery that we should use to define the measure of risk aversion in the case of relative lotteries. For this reason, the *Arrow-Pratt measure of relative risk aversion* is defined as $R^r(w) = -\frac{wu''(w)}{u'(w)} = wR^a(w)$. If an individual displays a greater value of relative risk aversion than another, then the former is more risk averse over relative lotteries than the latter.

The Arrow-Pratt measure of relative risk aversion shows up in a great many situations in microeconomic analysis, both in models of risk and uncertainty and in models of certainty. This is due to a simple fact, which can be noted by re-writing the measure of relative risk aversion in a slightly different way

$$R^r(w) = -\frac{wu''(w)}{u'(w)} = -\frac{w\left(\frac{du'(w)}{dw}\right)}{u'(w)} = -\frac{\left(\frac{du'(w)}{u'(w)}\right)}{\left(\frac{dw}{w}\right)}$$

So the measure of relative risk aversion is nothing more than the (negative of the) elasticity of marginal utility with respect to wealth.

Risk premium

Let's go back to absolute lotteries. In what we have done above, we always began with a situation of certainty, that is, our endowment points were risk-free. Now let's consider what can be done when we start off from a wealth distribution that involves risk, concretely we shall assume an endowment characterised by $w_1 > w_2$. In the same manner as previously, the indifference curve that passes through the endowment point defines the lower frontier of the acceptance set. This indifference curve cuts the certainty line at a point of wealth equal to, say, w^* in either state. w^* satisfies

$$u(w^*) = pu(w_2) + (1-p)u(w_1) \tag{3.5}$$

and it is known as the *certainty equivalent wealth*.

- **Exercise 3.7**. What is the certainty equivalent wealth for an individual with the lottery of the Saint Petersburg paradox, assuming that his utility function is $u(w) = \ln(w)$ and that his initial wealth (before the lottery prize is added) is the risk-free quantity 0?

- **Answer**. When the utility function is the logarithmic function $\ln(w)$, we know that the utility of the St. Petersburg paradox lottery is just $\ln(2)$. But the St. Petersburg paradox question is posed as if the bettor had no wealth other than what is obtained via the lottery. Thus, the certainty equivalent wealth for the lottery, under the assumption that the bettor has 0 wealth outside of the lottery, is the wealth of 2. Curiously then, when Bernoulli posed his solution to the paradox, he anticipated the concept of certainty equivalent wealth, but not the concept of willingness-to-pay for participating in the lottery.

Since the indifference curve is strictly convex, it is always true that $E\widetilde{w} = \overline{w} > w^*$. Indeed, the difference between the expected value and the certainty equivalent, $\overline{w} - w^* \equiv \pi$, gives us a second way to measure risk aversion. Clearly, $\pi = 0$ is possible only if the indifference curve is linear (it coincides with the iso-expected value line), that is, risk aversion is zero. Also, given an initial lottery, the more convex is the indifference curve (the greater is risk aversion), the lower will be w^*, and so the greater will be π. The variable[5] π is known as the *risk premium.*

- **Exercise 3.8**. Assume a strictly risk averse decision maker with a risky endowment such that his wealth is w_1 with probability $1-p$ and w_2 with probability p. Assume that $w_1 > w_2$. Write the equation that implicitly defines the risk premium as a function of w_1, w_2 and p. Use your equation to work out the value of the risk premium for the extreme points $p = 0$ and $p = 1$. Use this information to sketch a graph of how you think that the risk premium should look as a function of p. Now confirm mathematically whether or not the risk premium is convex or concave in p. Find the equation that characterises the turning point of the risk premium as a function of p. Draw a graph, with wealth on the horizontal axis and utility on the vertical, with a construction that indicates exactly this level of the risk premium.

- **Answer**. The equation that defines the risk premium (π) is

$$(1-p)u(w_1) + pu(w_2) = u(E\widetilde{w} - \pi)$$

where of course $E\widetilde{w} = (1-p)w_1 + pw_2$. When $p = 0$, the above equation would read $u(w_1) = u(w_1 - \pi)$, and this just says that with $p = 0$ we have $\pi = 0$. Likewise, with $p = 1$ the equation reads $u(w_2) = u(w_2 - \pi)$, which again implies $\pi = 0$. Since for any other p (i.e., for $0 < p < 1$) we have $\pi > 0$ due to risk aversion, you should sketch a graph that shows π as a concave function on the support [0,1], taking the value 0 at the two endpoints and taking positive values at all intermediate points. To confirm concavity of π in p we need to derive with respect to p the equation that defines the risk premium. Deriving once, we get

$$\begin{aligned} -u(w_1) + u(w_2) &= u'(E\widetilde{w} - \pi)\left(\frac{\partial(E\widetilde{w} - \pi)}{\partial p}\right) \\ &= u'(E\widetilde{w} - \pi)\left(\frac{\partial E\widetilde{w}}{\partial p} - \frac{\partial \pi}{\partial p}\right) \end{aligned}$$

[5] Actually rather than being a normal "variable" π is a function. In principle, it changes with any of the system's parameters.

Since $\frac{\partial E\widetilde{w}}{\partial p} = -w_1 + w_2$, this is just

$$-u(w_1) + u(w_2) = u'(E\widetilde{w} - \pi)\left(-w_1 + w_2 - \frac{\partial \pi}{\partial p}\right)$$

Deriving a second time we get

$$0 = u''(E\widetilde{w} - \pi)\left(-w_1 + w_2 - \frac{\partial \pi}{\partial p}\right)^2 - u'(E\widetilde{w} - \pi)\frac{\partial^2 \pi}{\partial p^2}$$

This says that

$$\frac{\partial^2 \pi}{\partial p^2} = \frac{u''(E\widetilde{w} - \pi)\left(-w_1 + w_2 - \frac{\partial \pi}{\partial p}\right)^2}{u'(E\widetilde{w} - \pi)} < 0$$

So indeed the risk premium is strictly concave in p.

The turning point is that at which $\frac{\partial \pi}{\partial p} = 0$. From the equation for the first derivative, write

$$\begin{aligned}
-u(w_1) + u(w_2) &= u'(E\widetilde{w} - \pi)(-w_1 + w_2) - u'(E\widetilde{w} - \pi)\frac{\partial \pi}{\partial p} \\
u'(E\widetilde{w} - \pi)\frac{\partial \pi}{\partial p} &= u'(E\widetilde{w} - \pi)(-w_1 + w_2) + u(w_1) - u(w_2) \\
\frac{\partial \pi}{\partial p} &= \frac{u'(E\widetilde{w} - \pi)(-w_1 + w_2) + u(w_1) - u(w_2)}{u'(E\widetilde{w} - \pi)}
\end{aligned}$$

So it turns out that we get $\frac{\partial \pi}{\partial p} = 0$ at the point where $u'(E\widetilde{w} - \pi)(-w_1 + w_2) + u(w_1) - u(w_2) = 0$. This point is better identified as the point such that

$$u'(E\widetilde{w} - \pi) = \frac{u(w_2) - u(w_1)}{w_2 - w_1}$$

Multiply the right-hand-side by -1 both in the numerator and the denominator, so that this reads $u'(E\widetilde{w} - \pi) = \frac{u(w_1) - u(w_2)}{w_1 - w_2}$.

The point in question is identified in Figure 3.11. It is the point at which the slope of the utility function is equal to that of the line joining the two points corresponding to the two options of wealth.

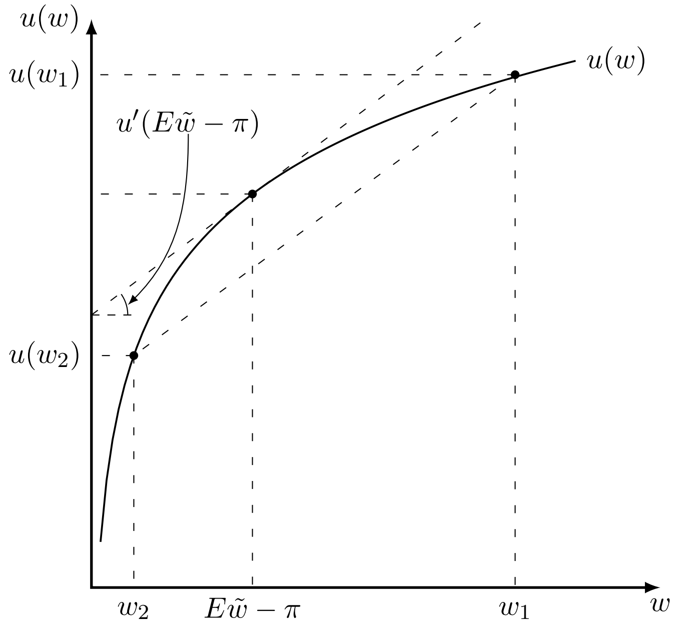

Figure 3.11 – Graphical construction of the maximum level of risk premium

Arrow-Pratt approximation

A logical thing to think about is exactly how the risk premium relates to the Arrow-Pratt measure of absolute risk aversion. To begin with, note that from the definition of the certainty equivalent, and from the definition of the risk premium, we can write

$$pu(w_2) + (1-p)u(w_1) = Eu(\widetilde{w}) = u(w^*) = u(\overline{w} - \pi)$$

Thus, the risk premium is the maximum amount of wealth that the individual would be willing to pay to substitute his lottery for the one with no risk at all but with the same expected value. It is now useful to split the initial lottery into two parts[6] – the risk-free part, w_0, and a risky part which we denote by the random variable $\widetilde{x}$, whose expected value is $E\widetilde{x} = \overline{x}$, multiplied by a constant, k. In this way, the endowed expected utility is

$$pu(w_0 + kx_2) + (1-p)u(w_0 + kx_1) = Eu(w_0 + k\widetilde{x})$$

[6]There is no loss of generalty here, since we can always have $w_0 = 0$.

We can now study risk-free situations by simply using $k = 0$.

Since the expected value of the initial situation is $E(w_0 + k\widetilde{x}) = w_0 + kE\widetilde{x} = w_0 + k\overline{x}$, the risk premium corresponding to the initial situation is defined by π such that

$$Eu(w_0 + k\widetilde{x}) = u(w_0 + k\overline{x} - \pi) \tag{3.6}$$

We shall study the behaviour of π as k changes, maintaining the rest of the parameters constant, and so it is easier if we write the risk premium as $\pi = \pi(k)$. Above all, we are interested in the function $\pi(k)$ around the point $k = 0$, that is, we are interested in small risks, in order to relate the risk premium with the absolute risk aversion function defined above, which you should recall is also defined around a point of certainty.

First, we take a second-order Taylor's expansion of $\pi(k)$ around the point $k = 0$:

$$\pi(k) \approx \pi(0) + k\pi'(0) + \frac{k^2}{2}\pi''(0) \tag{3.7}$$

In this equation, we are going to substitute for the values of $\pi(0)$, $\pi'(0)$ and $\pi''(0)$. We begin by noting that, if we set $k = 0$ in (3.6), then we directly obtain the result $\pi(0) = 0$. Second, derive (3.6) with respect to k to obtain

$$E\widetilde{x}u'(w_0 + k\widetilde{x}) = (\overline{x} - \pi'(k))u'(w_0 + k\overline{x} - \pi(k))$$

When $k = 0$, we get $u'(w_0)\overline{x} = (\overline{x} - \pi'(0))u'(w_0 - \pi(0)) = (\overline{x} - \pi'(0))u'(w_0)$ where we have used the fact that $\pi(0) = 0$. But then we must have $\overline{x} = \overline{x} - \pi'(0)$, and so $\pi'(0) = 0$. Deriving (3.6) a second time with respect to k yields

$$\begin{aligned} E\widetilde{x}^2u''(w_0 + k\widetilde{x}) = {} & (\overline{x} - \pi'(k))^2u''(w_0 + k\overline{x} - \pi(k)) \\ & - \pi''(k)u'(w_0 + k\overline{x} - \pi(k)) \end{aligned}$$

But since (as we have just seen) $\pi(0) = 0$ and $\pi'(0) = 0$, setting $k = 0$ gives us the result that

$$E\widetilde{x}^2u''(w_0) = \overline{x}^2u''(w_0) - \pi''(0)u'(w_0)$$

that is,

$$\pi''(0) = -\frac{u''(w_0)}{u'(w_0)}(E\widetilde{x}^2 - \overline{x}^2)$$

Finally, substitute these three results into (3.7) to obtain

$$\pi(k) \approx \frac{k^2}{2}\left(-\frac{u''(w_0)}{u'(w_0)}(E\widetilde{x}^2 - \overline{x}^2)\right) = \frac{k^2(Ex^2 - \overline{x}^2)}{2}R^a(w_0)$$

Now, from the definition of variance, simple steps that the reader can (and should) check, reveal that

$$\sigma^2(k\widetilde{x}) = E(k\widetilde{x} - k\overline{x})^2 = k^2(E\widetilde{x}^2 - \overline{x}^2)$$

and so we end up with the equation

$$\pi(k) \approx \frac{\sigma^2(k\widetilde{x})}{2}R^a(w_0) \tag{3.8}$$

Equation (3.8) is known as the *Arrow-Pratt approximation* to the risk premium. It shows how, as we have already indicated, the greater is the measure of absolute risk aversion, the greater is the risk premium, but it also clearly shows that for a given measure of risk aversion, the greater is the variance of the lottery, the greater is the risk premium.

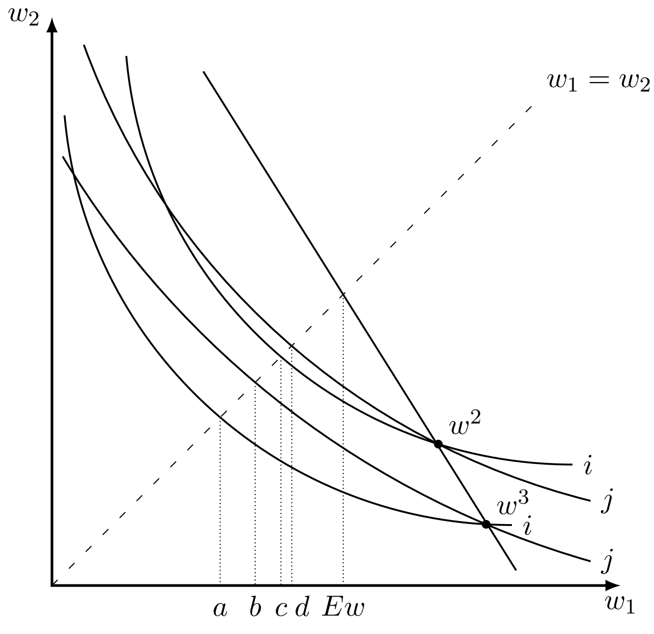

Figure 3.12 – Effect of greater risk aversion and greater risk upon the risk premium

In Figure 3.12 we have drawn two lotteries with the same expected value and different variances. The expected utility indifference curves through each of these two lotteries are also drawn for two individuals, one of whom (individual i) is more risk averse than the other (individual j). If we concentrate on either of the two different lotteries, then it is clear that the risk premium is greater for the individual with greater risk aversion, so the risk premium is increasing in risk aversion. On the other hand, if we concentrate on the indifference curves of any of the two individuals, then it is also clear that holding risk aversion constant and increasing variance also leads to an increase in the risk premium.[7]

3.4 Slope of risk aversion

It is important to note that the Arrow-Pratt absolute risk aversion function is a properly defined function (as is the relative risk aversion function), and so it is natural to wonder how risk aversion is affected by an increase in w. First, since $R^r(w) = wR^a(w)$, we get

$$R^{r\prime}(w) = R^a(w) + wR^{a\prime}(w)$$

Thus, always within the assumption that the utility function itself is increasing and concave (so that both relative and absolute risk aversion are positive) we can directly conclude that

1. If absolute risk aversion is not decreasing ($R^{a\prime}(w) \geq 0$), then relative risk aversion is increasing ($R^{r\prime}(w) > 0$).
2. If relative risk aversion is not increasing ($R^{r\prime}(w) \leq 0$), then absolute risk aversion is decreasing ($R^{a\prime}(w) < 0$).

Second, if we derive the definition of absolute risk aversion, and if we define $P(w) \equiv -\frac{u'''(w)}{u''(w)}$, then we obtain

$$\begin{aligned} R^{a\prime}(w) &= -\left(\frac{u'''(w)u'(w) - u''(w)u''(w)}{u'(w)^2}\right) \\ &= -\frac{u'''(w)}{u'(w)} + \left(\frac{u''(w)}{u'(w)}\right)^2 \\ &= -\frac{u'''(w)}{u'(w)}\frac{u''(w)}{u''(w)} + R^a(w)^2 \end{aligned}$$

[7]Actually, this may not hold risk aversion constant, as the certainty equivalent wealth changes, which may imply that risk aversion also changes for a particular individual.

$$= -P(w)R^a(w) + R^a(w)^2$$
$$= R^a(w)\left(R^a(w) - P(w)\right)$$

At the second step we can note that, $u'''(w) \geq 0$ is a necessary (but not sufficient) condition for decreasing absolute risk aversion; $R^{a\prime}(w) < 0$. In words, a necessary condition for decreasing absolute risk aversion is that marginal utility is convex. But we have already assumed that $u'(w) > 0$ and that $u''(w) < 0$ for all w, that is, marginal utility is positive and decreasing. From that, we can directly conclude that, at least for very large values of w, marginal utility will indeed be convex (if not, it would either have to be negative or increasing – draw a graph of marginal utility if you are not convinced).

At the final step, we can also conclude that a necessary *and sufficient* condition for decreasing absolute risk aversion is that $R^a(w) < P(w)$. The function $P(w)$ as defined above is known as "absolute prudence", and so absolute risk aversion is decreasing if (and only if) absolute risk aversion is less than absolute prudence. Another way of looking at prudence is to consider the utility function $v(w) = -u'(w)$. Prudence of $u(w)$ is then just the Arrow-Pratt measure of absolute risk aversion of $v(w)$. So $u(w)$ displays decreasing absolute risk aversion if $u(w)$ is less risk averse than is $-u'(w)$. The concept of prudence turns out to be important for decisions that involve savings as a hedge against risk, and it is normally accepted that risk averse individuals also display positive prudence, implying that indeed $u'''(w) > 0$. We study exactly this kind of problem in the next chapter.

In short, it is very often accepted that absolute risk aversion is in fact decreasing (indeed, a common assumption – which is also often found to correspond to real life choices in empirical analyses – is that relative risk aversion is constant). In graphical terms, decreasing absolute risk aversion corresponds to a family of indifference curves that become more and more linear as we move away from the origin of the graph.

- **Exercise 3.9**. Calculate the Arrow-Pratt measures of absolute risk aversion, relative risk aversion, and absolute prudence, for the following utility functions: $u(w) = \ln(w)$, $u(w) = -ae^{-bw}$, and $u(w) = -aw^2 + bw + c$, where a, b and c are all positive constants.

- **Answer**. It is easiest to do each of these simply by construction. That is, work out the first and second derivatives, divide the second derivative by the first and multiply by -1 to get absolute risk aversion. Multiply the absolute risk aversion by w to get relative risk aversion. Calculate prudence by working out the third derivative and dividing that by the second derivative (and, of course multiplying by -1). If you carry out each of these derivatives correctly, you should arrive at the following conclusions: (a) for the function $u(w) = \ln(w)$, absolute risk aversion is $\frac{1}{w}$, relative risk aversion is 1, and absolute prudence is $\frac{2}{w}$, (b) for the function $u(w) = -ae^{-bw}$, absolute risk aversion is b, relative risk aversion is bw, and absolute prudence is b, (c) for the function $u(w) = -aw^2 + bw + c$ absolute risk aversion is $\frac{2a}{b-2aw}$, relative risk aversion is $\frac{2aw}{b-2aw}$, and absolute prudence is 0. Thus, the logarithmic function is decreasing absolute risk aversion (DARA) and constant relative risk aversion (CRRA), the exponential function is constant absolute risk aversion (CARA) and increasing (actually linear) relative risk aversion (IRRA), and the quadratic function has increasing absolute risk aversion (IARA) and increasing relative risk aversion (IRRA).

Summary

In this chapter you should have learned the following:

1. A principal aspect of expected utility preferences is that they are linear in probabilities.
2. If the utility function for money is concave (second derivative negative), then the decision maker suffers what is known as "risk aversion". Risk aversion is a situation in which an increase in variance that leaves expected value unchanged leads to a less preferred outcome.
3. Risk aversion shows up in the Marschak-Machina triangle as indifference lines that are less steep than the iso-expected value lines. It shows up in the contingent claims environment as indifference curves that are convex.
4. The standard graphical environment for studying choice under risk is the contingent claims setting. In that setting, we represent the outcomes (prizes) of a lottery on the two axes. The probabilities of the outcomes then show up in the slopes of the indifference curves (marginal rates of substitution) and the slopes of expected value lines in the graph.

5. Different preferences are defined by different risk aversion. Risk aversion at any given level of wealth w can be measured by the function $R^a(w) = -\frac{u''(w)}{u'(w)}$, the Arrow-Pratt measure of absolute risk aversion. If two utility functions are related by a positive linear transformation, they will show the same level of risk aversion for every level of wealth, and so they show the same preferences exactly. But if one utility function is a concave transform of another, then the former is more risk averse than the latter.
6. Other important functions that are relevant to risk aversion are relative risk aversion, $R^r(w) = -\frac{wu''(w)}{u'(w)}$, and prudence $P(w) = -\frac{u'''(w)}{u''(w)}$. Relative risk aversion is a measure of the elasticity of marginal utility to wealth, and prudence is the absolute risk aversion of the utility function $v(w) = -u'(w)$.
7. The slope of absolute risk aversion is an important ingredient to many problems in economics. It is often assumed that absolute risk aversion is decreasing (decision makers are less risk averse the wealthier they become). This is equivalent to saying that absolute risk aversion is less than prudence.
8. Two other important concepts for decision making under risk are the certainty equivalent wealth and the risk premium corresponding to a given risk. Certainty equivalent wealth is the level of wealth that generates exactly the same level of utility as a given lottery, and the risk premium is the difference between the expected value of wealth and the certainty equivalent wealth.
9. We can estimate the value of the risk premium, at least for small risks, using the Arrow-Pratt approximation. Under this approximation, the risk premium is (approximately) equal to half of the variance of the lottery multiplied by the level of absolute risk aversion measured at the expected level of wealth. This approximation confirms that the risk premium increases with risk aversion and with the risk of the lottery (its variance).

Problems

1. Prove mathematically that a movement upwards along a line of constant expected value in the Marschak-Machina triangle corresponds to an increase in variance of wealth.
2. Use Jensen's inequality to prove that if $u(w)$ is concave, then the iso-expected value lines in the Marschak-Machina triangle are steeper than the indifference curves.

3. Assuming strictly risk averse preferences, indicate in a Marschak-Machina triangle a lottery, denoted by "lottery A", between only the best and worst prizes that is indifferent to receiving the intermediate prize for sure. Draw the indifference curve going through lottery A, and evaluate its slope in terms of probabilities. Indicate on the graph the set of lotteries that is at least as good as lottery A. Is this a convex set?
4. In a variant of the two lotteries in exercise 3.2, assume now that each trial of the lotteries pays \$1 with probability $(1-p)$ and $-\$1$ with probability p. Can the two lotteries implied by a single, and a repeated, trial of this be located in a single Marschak-Machina triangle? If p were equal to one-half, would you expect a risk averse bettor to accept a single trial of this game? How about the two trial version of the game? (Hint: try using Jensen's inequality for a concave utility function).
5. Use Jensen's inequality to prove that if $u(w)$ is concave in the scalar w, then $Eu(\widetilde{w})$ is concave in the vector (w_1, w_2).
6. Assume that an individual has risk-free wealth of \$350,000. Find the limit value of relative risk aversion for which the individual should certainly reject a bet that involves winning \$105 with probability one half and losing \$100 with probability one half? (Clue: use the Arrow-Pratt approximation for the risk premium).
7. Consider a two-state problem in which a strictly risk averse expected utility maximiser must allocate an initial wealth of w_0 over two states, where the state contingent claims (w_1 and w_2) are traded at prices q_1 and q_2 respectively. Assuming an interior solution, prove that

$$\frac{\partial w_1^*}{\partial w_0} = \frac{R^a(w_2^*)}{R^a(w_1^*)}\frac{\partial w_2^*}{\partial w_0}$$

where $R^a(w_i)$ is the Arrow-Pratt measure of absolute risk aversion in state i. What does this result imply for the signs of the effect of an increase in initial wealth on the demand for the two state contingent claims?
8. Continuing from problem 7, now derive the budget constraint, $w_0 = q_1 w_1^* + q_2 w_2^*$ with respect to initial wealth, and solve out for the values of $\frac{\partial w_i^*}{\partial w_0}$ for $i = 1, 2$. Assuming that absolute risk aversion is constant in wealth, how would an increase in absolute risk aversion affect your solution?

9. Draw a graph in contingent claims space that represents the indifference curves corresponding to constant absolute risk aversion. Be careful to clearly indicate how CARA shows up in the graph. Show that if two different points in the graph have the same marginal rate of substitution, then they must lie on a straight line with slope equal to 1 (i.e., they must both have the same variance).
10. Repeat your graph of the previous problem, but this time for the case of constant relative risk aversion, CRRA. This time, you need to show that if two points have the same marginal rate of substitution, then they must lie on a straight line that is a ray from the origin.
11. In exercise 3.9 we saw that the utility function $u(w) = -ae^{-bw}$ corresponds to constant absolute risk aversion. However, when both a and b are positive numbers, it also corresponds to negative utility. Do you think that negative utility is unreasonable for the analysis of choice? Explain why or why not.
12. Define the utility function $v(w) \equiv -u'(w)$. What are the signs of the first and second derivatives of this function? What is the absolute risk aversion of the function? Assuming that $u(w)$ displays decreasing absolute risk aversion, is $v(w)$ more or less risk averse than $u(w)$?
13. Prove that the set of utility functions that display decreasing absolute risk aversion is a convex set. Is the same true for the set of constant absolute risk aversion functions?