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# 1

# Economic building blocks and the importance of elasticities

## 1.1 Introduction

Some argue that economics stands out among the social sciences because of its precise and measurable nature – quite different (economists like to think) from many of the other social sciences, such as psychology and sociology. Indeed, the objects of economics – whether they be costs, market prices, quantities traded, or any other transaction between individuals – lend themselves to quantitative analysis. For instance, there are disparities in the observable costs of different goods and services, and these affect which items are in great demand and which are not. Take income, which varies between countries and individuals. Even when things cannot be expressed directly in monetary values, such as the damage inflicted by pollution, economists measure other quantities (for example, in tons of emissions) and propose monetary equivalents for compensating damages. Such quantification is possible because transactions between producers and consumers are measurable, at least in principle, which cannot be said of other social sciences.

In this chapter we identify the building blocks of the economic model that will be introduced in Chapter 2 and refined thereafter in subsequent chapters. Economics has many subdisciplines, including monetary economics, international trade, environmental economics, business economics, public economics, health economics and industrial organization, but a division is usually drawn between *micro*economics and *macro*economics. Their interrelationship is key to our approach in this book. Macro, for short, is concerned with economic aggregates, such as the level of unemployment, the price level, the quantity of money, and the rate of productivity growth. However, it has probably become abundantly clear during your studies that an understanding of these issues requires knowledge of microeconomics. For example, the price "level" is an average of many different prices in an economy – so we had better understand the formation and role of each of them before we start working with some weighted average. Microeconomics is the foundation of economics. It explains the magnitude of economic transactions using data describing the following three fundamentals: consumers' preferences, the limits of technology, and available physical resources. This three-way approach pervades all branches of economics and can even be applied beyond the discipline: the tools of microeconomic analysis are used in many interdisciplinary sciences, including ecology, marketing, and conflict analysis.

In microeconomics we study the behavior of interdependent consumers and producers. Consumers supply labor and capital and demand goods and services for consumption. Producers are the mirror image, demanding labor and capital and, in return, supplying produced goods and services. The fact that producers also use produced goods and services compounds the degree of interdependence. Moreover, many (if not all) produced goods and services, such as electricity, computers, buildings, and cleaning services, are used by both consumers and producers. The economy can be thought of as a remarkable machine that coordinates all these demands and supplies. An understanding of this machine requires a thorough knowledge of its constituent parts – consumers and producers – and the coordination mechanism. Like a car, an economy consists of many parts. But, rather than having a driver, an economy's parts – the correct supply of fluids, maintenance, and other goods and services – are coordinated by *markets*, where prices vary to match demand and supply.

It is particularly important to understand these interdependencies when addressing economic policy issues. For instance, economists advocate free trade even when it constitutes a threat to domestic businesses. The European Union is a free-trade zone, and free-trade agreements among non-member countries are an even more common example of threats to domestic industry. The negotiations that precede such arrangements involve scenario analyses that assess the pros and the cons of free trade. In order to understand which domestic products will be out-competed by imports and, conversely, which products will be produced in greater quantities for export to new markets, we must be able to determine the supplies and demands that will prevail under a particular free-trade arrangement. The new imports and new exports are interdependent because national economies pay for their imports with their exports. So we need an economic model to help us understand these relationships and determine whether or not citizens will be better off in the new situation (as economists suggest), and whether the new situation will be "efficient." In this book we extend this economic analysis to environmental policy. For example, we will see that the Netherlands has a relatively clean pattern of consumption but not of production. By exporting products, the Dutch import pollution. If the Dutch take over production from Germany, for instance, Dutch exports increase but, at the same time, pollution is relocated from Germany to the Netherlands. It is efficient if some countries, with relatively clean production, pollute more than others. Indeed, as one country's living standard gains from trade, the environment may gain from specialization in relatively clean production.

Interdependencies also play a role at a more micro level. When energy suppliers overcharge, not only do household electricity bills increase but other goods and services that must be purchased are also affected, since electricity is an input to all produced commodities and thus codetermines their prices. What is the right electricity price? This question must be answered on the basis of an efficiency criterion – namely, that citizens cannot be better off – and we will learn how to determine the efficient price. However, this is not easy. The prices of many goods and services cannot be determined without knowledge of the price of electricity. For instance, electricity is used to produce cables, but cables are used to produce electricity. We will develop a general equilibrium model that accommodates roundabout production. Roundaboutness is excluded from standard microeconomic textbooks but here it is an integral part of the discussion of microeconomic theory. The first main result is the construction of market-clearing equilibrium prices; the other two are welfare theorems. The First Welfare Theorem shows that the equilibrium allocation is efficient, while the second shows that any other efficient allocation can be brought about by a combination of wealth redistribution and the market mechanism.

Overcharging, an instance of the abuse of market power, raises a number of interesting questions. First and foremost, what is market power? In its pure form you may think it is monopoly power, unhindered by competition; but this is not true. A monopoly may actually have little power. For instance, some telephone services are provided by a single carrier, but, if the incumbent overcharges, a potential entrant may take over. Thus, we define monopoly power as the ability to raise price above cost. Here it is important to note that the determination of cost is also an issue that requires much information and, once again, an economic model. Second, is market power bad? In order to answer this question we must relate market power to inefficiency. Third, when should we be on the look-out for market power? A low number of firms in an industry is not a sufficient indicator of market power. Demand plays a significant role, too. If customers have substitutes it is difficult to exercise market power because elastic demands counter anti-competitive behavior. Last, but not least, the conduct of producers may be collusive. All these issues will be dealt with and quantified using an economic model that we will construct. We will see, for example, that at the turn of the millennium Spanish airlines appeared to compete, even though their rates were not much lower than those a cartel might have set. In the real world, other departures from efficient competitive outcomes are the rule, rather than the exception. From a policy point of view, it is interesting to measure the gap between price and cost – not only in price but also in terms of consumers' valuations. The distinctive feature of the model we develop later in this book is that all the key concepts are measurable.

## 1.2 What is an economy?

The analysis of equilibrium and efficiency presented here will be more formal than analysis you have seen before, and therefore we must be more formal in defining the subject of our analysis. As in any science, it is extremely important to clearly distinguish what is considered given and what has still to be explained. The variables that are considered given are called *exogenous* variables, and the variables that have to be explained are called *endogenous* variables. Economists analyze the behavior of consumers, with consumers' preferences for consumption goods, services, and leisure time, as well as consumers' initial resources, as givens. Resources include not only any goods they inherit but also the skills they were given at birth and by upbringing. All these are exogenous variables. Clearly, though, the consumers' preferences are not simply "given" but also are shaped by their exposure to advertisements and social interactions. Economists recognize this, but their focus is not on explaining *why* people have particular tastes: This is left to psychologists and cultural anthropologists.

Economists also analyze producers' behavior, using the technological possibilities available to them. Technology is also exogenous, though it is always changing and, thus, not a "given." For example, we cannot take for granted the possible methods of making power (for example, coal, oil, nuclear, or solar), since so much effort is directed toward expanding these possibilities. Therefore economists leave it to engineers to delineate the state of the art.

What economists consider to be given – the exogenous variables of preferences, resources, and technology – are the building blocks of an economy, or *fundamentals*. Through the interaction between households and firms (and possibly other actors), the fundamentals determine *outcomes*, which are the endogenous variables that we seek to explain. These include the quantities of all goods and services produced and consumed, as well as their prices. It is important to grasp the direction of economic reasoning, which moves from fundamentals to outcomes. Many people, including economists, are confused about this.

A simple example will help to illustrate the issue. A number of people have failed their driving test several times and have given up. Now let us ask them, "Which car would you prefer to drive, a Bentley or a Volkswagen?" Many students assume the answer would be "A Bentley" because that is the more valuable car. But why is it more valuable? The answer is that other people – those *with* a driving permit – prefer to drive a Bentley rather than a Volkswagen. Now imagine that the world is populated *only* with non-drivers. In such a world, each person would judge the two cars on the basis of the cars' intrinsic values; but, having no license to drive, they would be indifferent. Thus, a Bentley would be as valueless as a Volkswagen. So this "preference" for the Bentley is misleading because it is based on the car's market price. As microeconomists, we want to explain the car's market price on the basis of consumers' intrinsic preferences and resources, as well as the car's technology. If we make one of the fundamental economic building blocks – namely, consumers' preferences – dependent on something else we want to explain – that is, the endogenous variable, price – our analysis becomes a mess of circular reasoning. For this reason, we must define preferences *irrespective* of price on the basis of the physical characteristics of goods and services.

Sensible policymaking requires a systematic mode of economic reasoning. First and foremost, we need theory to sort out what we want to explain – such as the standard of living – and which variables we will use for our investigation – such as the available means of production or rules of behavior. Then we need an economic model to quantify these relationships. The model we use is particularly apt for competitive market settings and therefore serves as a benchmark for other settings, providing an efficiency measure for their possible underperformance. Such analysis enhances the quality of policy recommendations.

## 1.3 Building blocks of an economy

We can be still more explicit about the building blocks of an economy. To understand the fundamental ideas underlying the economists' view of the world, consider an economy with only two products, goods and services. (Of course, this narrow example can and will be expanded in the course of this book.) The goods and services are produced by means of capital and labor. Here capital refers to physical structures, such as buildings and machinery, while labor is the human input required to operate the capital.

The production function for goods $P^G$ in the equation $y_G = P^G(K_G, L_G)$ indicates the quantity of goods output that can be produced with input quantities of capital and labor employed in manufacturing. Similarly, we have the equation $y_S = P^S(K_S, L_S)$ for services. In a private property economy, households own the means of production, capital, and labor. They formulate their demand functions according to their endowments and preferences. For simplicity, let us assume that all households have the same preferences and can be represented by a single household whose preferences for goods and services are summarized by a utility function $U(x_G, x_S)$ that subjectively values the consumption levels of the goods and services produced. One well-known example of a utility function is $U(x_G, x_S) = x_G.x_S$, which presumes that positive quantities of both goods and services are essential (in other words, a zero amount of either would imply zero utility). Thus, the building blocks of this simplified, imaginary economy are preferences (that is, the utility function $U$), resources (that is, the total quantities of capital $K$ and labor $L$), and technology (that is, the production functions $P^G$ and $P^S$). Supply and demand functions can then be derived, and, in turn, will determine the prices and quantities that we may expect to emerge in any free market. In the economic system of the model, there are four markets: two factor markets – one for capital, another for labor – and two product markets – one for goods, another for services. Recall that a factor is an "ultimate input" – for example, any raw material, rather than an intermediate, "produced input," such as electricity. Any change in economic outcomes must be due to a change in one or more of these fundamentals.

**Example 1.1** The prices of services rise relative to the prices of goods. Health care, education, and other services get ever more expensive (ten Raa and Schettkat, 2001). One important cause is technical progress in manufacturing, where less and less input is needed to produce the same level of output, since innovations that introduce new technologies and modes of production organization, such as outsourcing, spur productivity growth. Economic analysis shows that such productivity gains can yield higher prices for services – seemingly in contradiction to the intuition that productivity gains are always cost-saving. This is because the productivity gains in manufacturing drive costs lower in those industries. *In comparison*, then, services' costs go up, since those industries do not benefit from such cost-saving productivity gains.

A consumer's demand function tells us how much of a good or service she wants to buy, given its price. Roughly speaking, consumer demand goes down when the price of a good or service goes up. However, the quantity demanded by a consumer depends not only on the price of the good or service but also on the prices of other goods and services. There are two main reasons for this. First, if there are inexpensive substitutes, demand for a good or service will be weak. This was the bleak situation Microsoft's Xbox faced in the market for game consoles, where the lower prices of Nintendo's Wii and Sony's PlayStation 3 dampened the demand for Xboxes. This effect may also work the other way. For example, the low prices of game consoles boost the demand for games. Thus, the "other" price has a negative impact if the other product is a substitute (like a competing console) but has a positive impact if it is a complement (like a game).

Another type of cross-price effect is more macroeconomic. If a consumer is considering buying a product when the prices of all other products are going up, he not only chooses the inexpensive product but must also restrict his other purchases simply because he becomes poorer (if he has a stable income). Thus, price increases depress total demand in an economy by reducing real purchasing power.

## 1.4 Elasticities

Economists group all these effects under the heading of elasticities. Using these, we can discuss the concept of demand for a good or service as a function of its own price, if we keep all other prices fixed. We denote the quantity demanded by $D(p)$, where $p$ is the price and $D$ the demand function, which is declining. It is important to distinguish between a function, such as $D$, and a value it may take, such as $D(p)$. Thus, the statement "demand goes down" could mean two different things. One interpretation is that nothing changes at the consumer end: Households have the same level of demand for the good or service, but the good or service becomes more expensive – for example, due to a disruption in production. In this case, the demand function remains the same, but its value is reduced. Another interpretation is that the users of the good or service are no longer interested in it – perhaps their taste has changed and they assign less utility to it. In this case the demand function *has* changed.

Before we analyze elasticities formally, let us review some mathematical concepts that we will use extensively. The first is the *inverse* of a function. If the quantity demanded is

$$q = D(p) \tag{1.1}$$

we can solve equation (1.1) to get

$$p = P(q) = D^{-1}(q) \tag{1.2}$$

The inverse of a demand function $D$, function $D^{-1}$ in equation (1.2), is called the *willingness-to-pay* function – also denoted by $P$. Figure 1.1 depicts an example that reveals the price at which a given quantity of goods or services will be purchased.

A simple example of a demand function is the *linear* function $D(p) = \underline{q} - ap$, where $\underline{q}$ is the maximum quantity demanded (attained when the price is zero), and $a$ is the slope of the demand function. If the price goes up by 1€, demand is reduced by $a$ units (for example, liters of petrol) and the value of this function is reduced. If the maximum quantity demanded, $\underline{q}$, decreases, then the entire function changes, causing an inward shift. The graph of the function is a straight line with vertical intercept $\underline{q}$ and horizontal intercept $\underline{q}/a$. Here, $\underline{q}/a$ is the maximum price, at which demand becomes zero. If we solve the equation $q = \underline{q} - ap$, we obtain consumers' willingness to pay, $p = (\underline{q} - q)/a = P(q)$.

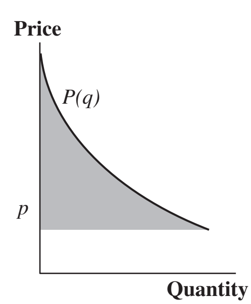


**Figure 1.1** The willingness-to-pay function.

Another example of a demand function is the *power* function $D(p) = Ap^{-\alpha}$, which has the advantage that it is defined for all positive prices. (In contrast, the linear demand function is defined only for prices below the maximum price $p \leq \underline{q}/a$, otherwise the quantity demanded would be negative.) In the linear demand function, consumers' responsiveness to price is given by the slope $a$. For example, if $a$ equals 2, then the demand for petrol is reduced by 2 liters if the price goes up by 1€. Formally, $\frac{dD(p)}{dp} = -a$, where the left-hand side denotes the *derivative* of the function $D$. It may also be denoted briefly by $D'(p)$.

In general, the derivative of a function measures the change in the dependent variable (in this case, the quantity demanded) per unit of change in the independent variable (that is, price) for small changes. It is defined by the following equation:

$$D'(p) = \frac{dD(p)}{dp} = \lim_{\Delta p \to 0} \frac{D(p + \Delta p) - D(p)}{\Delta p} \tag{1.3}$$

In equation (1.3) lim is short for limit, and the subscript $\Delta p \to 0$ means that the price change, $\Delta p$, is small – in fact, going toward zero. Thus, it is easy to see that the derivative of the sum of two functions is the sum of the derivatives of the two functions. This observation is useful when collecting individual demands to calculate overall market demand. In a linear demand function, the size of the price change is immaterial because the quotient equals the slope, $-a$, as equation (1.4) shows:

$$\begin{aligned} \frac{dD(p)}{dp} &= \lim_{\Delta p \to 0} \frac{D(p + \Delta p) - D(p)}{\Delta p} = \lim_{\Delta p \to 0} \frac{\underline{q} - a(p + \Delta p) - (\underline{q} - ap)}{\Delta p} \\ &= \lim_{\Delta p \to 0} \frac{-a\Delta p}{\Delta p} = -a \end{aligned} \tag{1.4}$$

However, measuring the responsiveness of demand by the reduction in liters per euro price increases is problematic. First, this measure of the responsiveness of demand depends on the choice of units. If we were to measure petrol in gallons – as Americans do – rather than the much smaller liter, this measure of the responsiveness of demand would shoot down. Similarly, if we use a weaker currency, like the American dollar, this measure of the responsiveness of demand would also be reduced. This is because the reduction in demand in response to a price increase of one US penny is smaller than that in response to an increase by one eurocent. Second, the relationship between demand and price depends on the unit of time. If we are talking about changes in weekly demand, then fluctuation in demand of two liters per week seems reasonable. However, if we are talking about changes in daily demand, this response would be quite strong.

Measuring such changes as percentages avoids such ambiguities. The *elasticity* of demand measures the *percentage* change in quantity per *percentage* change in price. It is defined by the following expression:

$$\varepsilon^D(p) = \frac{dD(p)/D(p)}{dp/p} \tag{1.5}$$

In this expression we use only proportionate changes, since the percentage conversion factors of 100 cancel out in the numerator and denominator. For example, if a price increase from 2 euros per liter to 2.01 euros per liter yields a demand reduction from 10 to 9.8 liters, then the elasticity expression (1.5) becomes $\frac{-0.2/10}{0.01/2} = -20\left(\frac{2}{10}\right) = -4$. This means that a 1 percent price increase leads to a 4 percent reduction in the quantity demanded.

While linear functions have constant derivatives, power functions have constant elasticities. To appreciate this mathematical fact, we must first take the derivative of a power function:

$$\frac{dAp^{-\alpha}}{dp} = -\alpha A p^{-\alpha-1} \tag{1.6}$$

Rule (1.6) shows that the derivative of a power function is obtained by multiplying the function with the power and then lowering the power in the exponent by one. This rule is a consequence of the definition of the derivative of a function, given in formula (1.3). Next we illustrate rule (1.6) for $\alpha = 2$:

$$\begin{aligned}
\frac{dD(p)}{dp} &= \lim_{\Delta p \to 0} \frac{D(p+\Delta p) - D(p)}{\Delta p} = \lim_{\Delta p \to 0} \frac{A(p+\Delta p)^{-2} - Ap^{-2}}{\Delta p} \\
&= \lim_{\Delta p \to 0} A \frac{1/(p+\Delta p)^2 - 1/p^2}{\Delta p} = \lim_{\Delta p \to 0} \frac{A}{\Delta p} \frac{p^2 - (p+\Delta p)^2}{p^2(p+\Delta p)^2} \\
&= \lim_{\Delta p \to 0} \frac{A}{\Delta p} \frac{-2p\Delta p - (\Delta p)^2}{p^2(p+\Delta p)^2} = \lim_{\Delta p \to 0} A \frac{-2p - \Delta p}{p^2(p+\Delta p)^2} \\
&= A \frac{-2p}{p^2 p^2} = -2Ap^{-3}
\end{aligned} \tag{1.7}$$

Substituting the result of (1.7) into (1.5) reduces the latter expression for the power function $D$, defined by $D(p) = Ap^{-\alpha}$, to simply:

$$\varepsilon^D(p) = \frac{dD(p)/D(p)}{dp/p} = \frac{dD(p)}{dp} \frac{p}{D(p)} = -\alpha A p^{-\alpha-1} \frac{p}{Ap^{-\alpha}} = -\alpha \tag{1.8}$$

**Review exercise** Show that a linear demand function – for example, $D(p) = 1 - p$ – has a variable elasticity.

As for linear functions, the coefficients signify the slope or derivative of the function, while (1.8) shows that for power functions the power signifies the elasticity. Since power functions $D(p) = Ap^{-\alpha}$ feature elasticity $\alpha$ at all prices, they are also called *constant elasticity* functions. For such functions, a price increase of 1 percent prompts a quantity decrease of $\alpha$ percent. The greater the elasticity (in absolute value), the more sensitive demand is to price changes. If $\alpha$ is greater than 1 (unity), demand is called *elastic*. If $\alpha$ is less than unity, demand is called *inelastic* because the percentage change in quantity is smaller in magnitude than the percentage change in price. The borderline case, $D(p) = Ap^{-1}$, is called *unitary elastic*. In such a case, which forms the "border" between elastic and inelastic demands, a 1 percent increase in price prompts a 1 percent reduction in demand.

It is easy to show that for elastic demand a price increase reduces revenue. Revenue is the product of price and quantity, hence $pAp^{-\alpha} = Ap^{1-\alpha}$ for constant elasticity demand functions. Indeed, revenue is a decreasing function if the exponent is negative, $1 - \alpha < 0$ or $\alpha > 1$, as it is when demand is elastic. If $\alpha < 1$ (inelastic demand), revenue is increasing. Therefore when demand is inelastic, a price increase raises revenue. These results remain valid for demand functions with varying elasticities.

**Review exercise** Determine the price range where demand $D(p) = 1 - p$ is elastic.

If the demand for a product is elastic, price changes induce relatively large changes in the quantity demanded by consumers. This fact has many important implications. For one, in such industries, firms have less market power. If they want to raise price above cost, they erode their own market. When consumers respond strongly to a price increase, firms think twice before doing so. For another, we will see that goods with low elasticities are the best candidates for taxes, since they will impact little on the quantity and, therefore, minimize the distortion of the economy. Land is the classic example because its quantity is essentially fixed. Indeed, Henry George (1966) has argued that land rent should be the *sole* source of government funding.

With their universal units of measurement, elasticities are practical and important in assessing the impact of price changes due to taxation or market power. In principle they may vary from point to point, since linear demand functions have a constant slope but a varying elasticity. However, many applied economic models avoid this complication by working with functions that feature a constant elasticity. We use such functions to illustrate the analysis throughout this book.

## 1.5 Positive and normative economics

It is crucial to distinguish between economic policy questions about what *should* happen and explanations of what *actually* does happen. Let us first discuss the latter. Given the fundamentals of an economy – preferences, resources, and technology – we will explain how resources are allocated among productive activities and who gets the fruits of these labors. Our explanation will cover the pricing of resources and products, the quantities transacted, and the levels of well-being achieved. Roughly speaking, people who are well endowed with resources that are critical to the production of desired goods and services will be well off. In this way, market economies reinforce inequities.

The branch of economic analysis that examines these issues is called *positive* economics. Its main tool is equilibrium analysis; as the word suggests, "equilibrium" corresponds to a state of rest, where there are no forces to instigate a change from the current position. By using equilibrium analysis in microeconomics, we will find the prices at which supply equals demand. Of course, an exogenous shock to the system – that is, a market or set of markets – will change the equilibrium. Microeconomic analysis includes studying the directions and magnitudes of changes in the outcome – that is, endogenous variables – in response to changes in exogenous variables. This type of analysis, called "comparative statics," compares the properties of different equilibria without necessarily revealing anything about the process of adjustment from one equilibrium to another. For instance, after a shock to the economy there will be a new equilibrium, but we cannot say if the economy adjusts to this by dipping once, twice, or not at all (at least not right away).

In principle, positive economics is neutral about assessing the quantities produced and consumed in equilibrium. Equilibrium may seem like a desirable state of affairs (after all, who would object to a demand for labor that meets supply – that is, full employment?). However, this is not necessarily the case. For example, if a firm has market power, as Microsoft does in desktop computer operating systems, the price of its product may be well above cost, and many potential customers who cannot afford the price but would be willing to pay the production cost are not served. These potential customers do not contribute to market demand. Thus, there is equilibrium between supply and demand at the higher price, but there is also latent demand that is not satisfied, but that could be met if the cost were covered. This is a hidden form of inefficiency.

The other branch of economics that studies what should be done in an economy is aptly called *normative* economics. There are two related sub-branches of normative economics. One investigates the inefficiency of a given allocation of resources across industries and of each industry's products among end users. The reasoning that led us to conclude that there is inefficiency in the market for computer operating systems is an example of the application of this branch of economics. In this case, the outcome used to evaluate efficiency is determined through the working of markets, with prices playing a role in allocation – that is, the number of computers purchased by consumers.

The other sub-branch of normative economics maps all the possible efficient allocations for an entire economy. It is important to understand that this does not require prices but, rather, involves mapping all the possible allocations of physical resources and products and then evaluating them on the basis of consumers' preferences. The range of such allocations is huge. For example, one allocation would put the bulk of one country's products in the hands of its president, though all products could just as well be in the hands of any other citizen. Prices may play a role in such allocations, but it is an indirect role. More precisely, we will investigate if any of these efficient allocations can be brought about by the market mechanism after a suitable redistribution of the resources.

Granted, there are "free-market economists" who equate equilibrium outcomes with efficient allocations. This is not a matter of dogma but depends on conditions that must be fulfilled, such as the absence of market power and external effects, like pollution. Microeconomics details the conditions under which equilibrium outcomes are efficient, and alternative efficient outcomes can be generated by the market mechanism. These results alert us to situations where these conditions are *not* fulfilled and aid us in designing policies that remedy the inefficiencies. One such policy is the relatively high taxation of items whose supply or demand is inelastic. In Chapter 2 we will analyze a simple economy without distortions caused by market power and external effects (that is, externalities). In other words, the close connection between equilibrium and efficiency holds for the simple economy presented in Chapter 2. The truth of this link will be ascertained only in later chapters, but we will make use of it right away to determine equilibrium prices by analyzing a simple efficiency problem.

## 1.6 Mathematical tools

In applying economic reasoning, the use of mathematical tools permits precision in defining concepts such as equilibrium and efficiency. Mathematics also helps us to make precise comparisons of different equilibrium outcomes. In addition to accurately determining the direction of changes in equilibrium outcomes, mathematical modeling allows us to quantify such effects. Ultimately, economic decision-makers – whether in households, firms, or governments – need quantitative estimates. For instance, estimates enable a household to plan a reallocation of spending in response to a price increase in petrol. Such estimates also help a firm decide how much to increase the price it charges consumers when its costs go up, or help a government to decide what tax rates to set for various goods and services.

Functions with a constant elasticity, such as the demand function $q = D(p) = Ap^{-\alpha}$, are especially amenable for estimation and therefore are commonly employed in economic modeling. This is because $q = Ap^{-\alpha}$ is *log-linear*; if we take the natural logarithm we transform it into the following linear relationship $\ln q = \ln(Ap^{-\alpha}) = \ln A - \alpha \ln p$. Consequently, elasticity is the slope of the regression line between the log of the quantity and the log of the price. This simple observation, which is easily extendable to multi-product markets, production, and other functions, explains why elasticities are frequently estimated in economics. Of course not all demand functions have constant elasticity, and in these cases a more complex function must be hypothesized. For example, such a demand function might be $\ln q = \ln A - \alpha \ln p + \beta(\ln p)^2$. Other functional forms are also possible. The key idea is that parameters, such as elasticities, allow us to test a particular mathematical hypothesis about the functional form against actual data. The resulting estimated parameters can subsequently be used in practical decision-making.

Before closing this chapter, three useful mathematical tools – the product rule, the quotient rule, and the chain rule – and the inverse operation of differentiation must be presented. The product rule applies to the product of two functions of a common variable. For example, if a firm sets price $p$, meets demand $D(p)$, and obtains a net price (excluding taxes) $N(p)$, then revenue is $N(p)D(p)$. How sensitive is revenue to price? In other words, what is the derivative of this product function? According to the *product rule*, it is the product of the derivative of the first function and the second function, plus the opposite product:

$$N(p)D(p) \text{ has derivative } N'(p)D(p) + N(p)D'(p) \tag{1.9}$$

The product rule given in (1.9) is a direct consequence of the fact that the derivative of $N(p)D(p)$ can be rewritten as (1.10):

$$\begin{aligned} &\lim_{\Delta p \to 0} \frac{N(p+\Delta p)D(p+\Delta p) - N(p)D(p)}{\Delta p} \\ &\quad = \lim_{\Delta p \to 0} \left[ \frac{N(p+\Delta p) - N(p)}{\Delta p} D(p+\Delta p) + N(p) \frac{D(p+\Delta p) - D(p)}{\Delta p} \right] \end{aligned} \tag{1.10}$$

The *quotient rule* has a slightly more complicated reading:

$$N(p)/D(p) \text{ has derivative } \frac{N'(p)D(p) - N(p)D'(p)}{D(p)^2} \tag{1.11}$$

The proof of rule (1.11) is as easy as that of the product rule:

$$\begin{aligned} &\lim_{\Delta p \to 0} \frac{N(p+\Delta p)/D(p+\Delta p) - N(p)/D(p)}{\Delta p} \\ &\quad = \lim_{\Delta p \to 0} \frac{[N(p+\Delta p) - N(p)]D(p) - N(p)[D(p+\Delta p) - D(p)]}{\Delta p D(p) D(p+\Delta p)} \end{aligned} \tag{1.12}$$

The *chain rule* is relevant whenever two functions are composed. For example, let a firm set price $p$, meet demand $q = D(p)$, and incur cost $C(q)$. It follows that cost is eventually a function of price, which is given by the composite function $C(D(p))$. How sensitive is cost to price? In other words, what is the derivative of this composite function? According to the chain rule, this is the product of the derivatives:

$$C[D(p)] \text{ has derivative } C'[D(p)]D'(p) \tag{1.13}$$

The proof of the chain rule (1.13) is straightforward. The derivative of $C[D(p)]$ is:

$$\begin{aligned} &\lim_{\Delta p \to 0} \frac{C[D(p+\Delta p)] - C[D(p)]}{\Delta p} = \\ &\lim_{\Delta p \to 0} \frac{C[D(p+\Delta p)] - C[D(p)]}{D(p+\Delta p) - D(p)} \cdot \frac{D(p+\Delta p) - D(p)}{\Delta p} \end{aligned} \tag{1.14}$$

If we substitute $q = D(p)$ and $\Delta q = D(p + \Delta p) - D(p)$, then the first factor on the right-hand side of equation (1.14) reads $\frac{C(q+\Delta q)-C(q)}{\Delta q}$. Since $\Delta q$ tends toward zero, as $\Delta p \to 0$, the proof of the chain rule is complete.

Typically, the quantity $q$ is large, and therefore $\Delta q = 1$ is small – that is, relatively close to zero. This explains why the derivative of cost, $\frac{C(q+\Delta q)-C(q)}{\Delta q}$ $(\Delta q \to 0)$, is closely approximated by $C(q+1) - C(q)$, the so-called *marginal cost* of the last unit. This explains why economists write *MC* instead of *C*'.

One simple, but practical, application of the chain rule is that $dD(\bar{p} - p)/dp = -D'(\bar{p} - p)$, where $\bar{p}$ is a large given price (otherwise $\bar{p} - p$ would be negative and $D(\bar{p} - p)$ undefined). The composite function consists of the linear function $\bar{p} - p$ and function $D$. The derivative of this linear function is $-1$, which explains the minus sign.

The second application is the rule for the derivative of an inverse. For instance, the inverse of the demand function given in equation (1.2) has the following derivative:

$$D^{-1}(q) \text{ has derivative } 1/D'(p) \tag{1.15}$$

Rule (1.15) states that the derivative of the inverse is the reciprocal of the derivative. The proof is simple. If we apply the inverse function (that is, the willingness-to-pay function) to the quantity $q = D(p)$, we obtain price $p$: $D^{-1}[D(p)] = p$. If we apply the chain rule to the left-hand side of the function, differentiation yields $(D^{-1})'[D(p)] \cdot D'(p) = 1$, while division yields $(D^{-1})'(q) = 1/D'(p)$.

The inverse operation of differentiation is integration. We define the concept of an integral by means of an area and then relate it to differentiation. Figure 1.1

reveals an interesting economic phenomenon – namely, that for the initial units of a good consumers' willingness to pay exceeds the price. (Think of how valuable the first liters of petrol you put in your car's tank are to you.) Consumers thus benefit from the law of one price. They buy up to the point where their willingness to pay equals the price. But since that same price also applies to all the other units bought, consumers pay less than their willingness to pay for those units. The difference between the two, called *consumer surplus*, is measured by the shaded area in Figure 1.1. The area under a nonnegative function $D$ between lower bound $a$ and upper bound $b$ is called the *integral* of $D$ between $a$ and $b$ and is denoted by $\int_a^b D(p)dp$, where the symbol $\int$ stands for the integral. Consumer surplus is given by $\int_p^\infty D(p')dp'$, where $\infty$ is infinity. If we differentiate the integral of $D$ with respect to the upper bound, applying the definition (1.3), we obtain the function we integrated:

$$\frac{d}{db}\int_a^b D(p)dp = \lim_{\Delta b \to 0} \frac{\int_a^{b+\Delta b} D(p)dp - \int_a^b D(p)dp}{\Delta b} = \lim_{\Delta b \to 0} \frac{\int_b^{b+\Delta b} D(p)dp}{\Delta p}$$

$$= \lim_{\Delta b \to 0} \frac{D(b)\Delta b}{\Delta b} = D(b) \tag{1.16}$$

Result (1.16) confirms that integration is differentiation in reverse and, moreover, is the key to the calculation of areas. (Similarly, differentiation with respect to the lower bound yields $-D(a)$.) For example, if the function is a power function, $D(b) = b^{-\alpha}$, it is the derivative of $b^{1-\alpha}/(1-\alpha) + c$ ($c$ is any constant), as is confirmed by subjecting the latter to formula (1.6). This is the so-called *primitive* function of $D(b) = b^{-\alpha}$, denoted $\int D$. This gives us $\frac{d}{db}\left(\frac{b^{1-\alpha}}{1-\alpha} + c\right) = D(b)$. Comparison of this with equation (1.16) shows that the area is $\int_a^b D(p)dp = \frac{b^{1-\alpha}}{1-\alpha} + c$. Since the constant is determined by substituting $b = a$, the area between $a$ and $b = a$ must be zero, hence $c = -\frac{a^{1-\alpha}}{1-\alpha}$. It follows that $\int_a^b D(p)dp = \frac{b^{1-\alpha}}{1-\alpha} - \frac{a^{1-\alpha}}{1-\alpha}$.

The procedure can be summarized as follows. To find the area under a nonnegative function $D$ between a lower and upper bound, find the primitive, evaluate it in the upper and lower bounds, and subtract the evaluation of the lower bound from that of the upper bound.

Since integration is differentiation in reverse, the integral inherits several properties from the derivative. For example, the integral of the sum of two functions must be the sum of the integrals. Thus the product rule (1.9) also has a counterpart. Since the derivative of $N(p)D(p)$ is $N'(p)D(p)+N(p)D'(p)$, we have $N(p)D(p)=\int[N'(p)D(p)dp+N(p)D'(p)]dp$. Hence, the integral of $N'(p)D(p)+N(p)D'(p)$ between $a$ and $b$ is $N(b)D(b)-N(a)D(a)$ or, rearranging one term,

$$\int_a^b N'(p)D(p)dp = -\int_a^b N(p)D'(p)dp + N(b)D(b) - N(a)D(a) \quad (1.17)$$

Formula (1.17) is called *integration by parts*. When $N(p)D'(p)$ is easier to integrate than a given $N'(p)D(p)$, this is a practical rule. When integrating the product of two factors, first try to find the primitive of one factor, then differentiate the other, and see if it simplifies.

## 1.7 Summary

The building blocks, or fundamentals, of an economy are consumer preferences (represented by utility functions), resources (that is, capital and labor), and technology (represented by production functions). In a private property economy, households own the resources and formulate their demand functions by maximizing utility, although they are subject to what their resources can buy. Their demands for goods and services depend on prices. Elasticities measure how changes in price affect demand. Producers use supply functions to identify which items are most profitable to produce. Supplies of goods and services also depend on prices. Positive economics seeks to explain the prevailing prices and the quantities bought and sold, as well as how sensitive these outcomes are to changes in the fundamentals of an economy. Normative economics detects inefficiencies – that is, demands that could be met but are not – and maps all possible efficient allocations of goods and services.

In addition to the concept of elasticity, this chapter discussed other useful tools – namely, the inverse function (that is, the willingness-to-pay function, if the underlying function is a demand function), differentiation (including the product rule, the quotient rule, and the chain rule), and the inverse operation of differentiation, integration (including integration by parts).

We are now equipped to analyze an economy, and that is just what we will do in the next chapter – albeit for a simple economy that involves only capital, labor, one good, and one service.

## Exercises

1. Are the following statements examples of positive or normative economics?
   a. In an economic crisis real wages (that is, wages adjusted for the cost of living) are low.

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b. If students pay for their education, less teachers' time is wasted.
c. Perfectly competitive economies perform better than autocratic ones.
d. Free-market economies generate inequality.

2. Consider a constant elasticity demand function $q = D(p) = Ap^{-\varepsilon}$ with elastic demand and a linear cost function $C(q) = cq$. Show price $p = c\varepsilon/(\varepsilon - 1)$ maximizes the difference between revenue and cost – that is, profit.
3. Which commodity has more elastic demand – housing or books? Why?
4. Show the price in exercise 2 is a decreasing function of elasticity and explain this phenomenon.
5. Show the consumer surplus of a constant elasticity demand function with elastic demand is $Ap^{1-\varepsilon}/(\varepsilon - 1)$.

## References

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