# Contents

Preface to the first edition ........................................... xviii
Preface to the second edition ....................................... xix
Preface to the third edition ......................................... xix
Preface to the fifth edition ......................................... xx
Hints on using the book ............................................. xxi
Useful background information ...................................... xxii

<table>
<thead>
<tr>
<th>Programme 1</th>
<th>Numerical solutions of equations and interpolation</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Learning outcomes</td>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>The Fundamental Theorem of Algebra</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Relations between the coefficients and the roots of a polynomial equation</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>Cubic equations</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>Transforming a cubic to reduced form</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>Tartaglia's solution for a real root</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>Numerical methods</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Bisection</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Numerical solution of equations by iteration</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>Using a spreadsheet</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>Relative addresses</td>
<td>14</td>
</tr>
<tr>
<td></td>
<td>Newton–Raphson iterative method</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>Tabular display of results</td>
<td>16</td>
</tr>
<tr>
<td></td>
<td>Modified Newton–Raphson method</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td>Interpolation</td>
<td>24</td>
</tr>
<tr>
<td></td>
<td>Linear interpolation</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>Graphical interpolation</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>Gregory–Newton interpolation formula using forward finite differences</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>Central differences</td>
<td>31</td>
</tr>
<tr>
<td></td>
<td>Gregory–Newton backward differences</td>
<td>33</td>
</tr>
<tr>
<td></td>
<td>Lagrange interpolation</td>
<td>34</td>
</tr>
<tr>
<td>Revision summary</td>
<td>38</td>
<td></td>
</tr>
<tr>
<td>Can you? Checklist</td>
<td>41</td>
<td></td>
</tr>
<tr>
<td>Test exercise</td>
<td>42</td>
<td></td>
</tr>
<tr>
<td>Further problems</td>
<td>43</td>
<td></td>
</tr>
<tr>
<td>Programme 2</td>
<td>Laplace transforms 1</td>
<td>46</td>
</tr>
<tr>
<td>--------------</td>
<td>----------------------</td>
<td>----</td>
</tr>
<tr>
<td><strong>Learning outcomes</strong></td>
<td>46</td>
<td></td>
</tr>
<tr>
<td>Introduction</td>
<td>47</td>
<td></td>
</tr>
<tr>
<td>Laplace transforms</td>
<td>47</td>
<td></td>
</tr>
<tr>
<td>Theorem 1</td>
<td>The first shift theorem</td>
<td>54</td>
</tr>
<tr>
<td>Theorem 2</td>
<td>Multiplying by $t$ and $t^n$</td>
<td>55</td>
</tr>
<tr>
<td>Theorem 3</td>
<td>Dividing by $t$</td>
<td>57</td>
</tr>
<tr>
<td>Inverse transforms</td>
<td>60</td>
<td></td>
</tr>
<tr>
<td>Rules of partial fractions</td>
<td>61</td>
<td></td>
</tr>
<tr>
<td>The ‘cover up’ rule</td>
<td>67</td>
<td></td>
</tr>
<tr>
<td>Table of inverse transforms</td>
<td>68</td>
<td></td>
</tr>
<tr>
<td>Solution of differential equations by Laplace transforms</td>
<td>69</td>
<td></td>
</tr>
<tr>
<td>Transforms of derivatives</td>
<td>70</td>
<td></td>
</tr>
<tr>
<td>Solution of first-order differential equations</td>
<td>72</td>
<td></td>
</tr>
<tr>
<td>Solution of second-order differential equations</td>
<td>74</td>
<td></td>
</tr>
<tr>
<td>Simultaneous differential equations</td>
<td>81</td>
<td></td>
</tr>
<tr>
<td><strong>Revision summary 2</strong></td>
<td>86</td>
<td></td>
</tr>
<tr>
<td><strong>Can you? Checklist 2</strong></td>
<td>89</td>
<td></td>
</tr>
<tr>
<td><strong>Test exercise 2</strong></td>
<td>90</td>
<td></td>
</tr>
<tr>
<td><strong>Further problems 2</strong></td>
<td>90</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Programme 3</th>
<th>Laplace transforms 2</th>
<th>92</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Learning outcomes</strong></td>
<td>92</td>
<td></td>
</tr>
<tr>
<td>Introduction</td>
<td>93</td>
<td></td>
</tr>
<tr>
<td>Heaviside unit step function</td>
<td>93</td>
<td></td>
</tr>
<tr>
<td>Unit step at the origin</td>
<td>94</td>
<td></td>
</tr>
<tr>
<td>Effect of the unit step function</td>
<td>94</td>
<td></td>
</tr>
<tr>
<td>Laplace transform of $u(t - c)$</td>
<td>97</td>
<td></td>
</tr>
<tr>
<td>Laplace transform of $u(t - c) \cdot f(t - c)$ (the second shift theorem)</td>
<td>98</td>
<td></td>
</tr>
<tr>
<td>Differential equations involving the unit step function</td>
<td>108</td>
<td></td>
</tr>
<tr>
<td>Convolution</td>
<td>112</td>
<td></td>
</tr>
<tr>
<td>The convolution theorem</td>
<td>117</td>
<td></td>
</tr>
<tr>
<td><strong>Revision summary 3</strong></td>
<td>118</td>
<td></td>
</tr>
<tr>
<td><strong>Can you? Checklist 3</strong></td>
<td>120</td>
<td></td>
</tr>
<tr>
<td><strong>Test exercise 3</strong></td>
<td>120</td>
<td></td>
</tr>
<tr>
<td><strong>Further problems 3</strong></td>
<td>121</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Programme 4</th>
<th>Laplace transforms 3</th>
<th>123</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Learning outcomes</strong></td>
<td>123</td>
<td></td>
</tr>
<tr>
<td>Laplace transforms of periodic functions</td>
<td>124</td>
<td></td>
</tr>
<tr>
<td>Periodic functions</td>
<td>124</td>
<td></td>
</tr>
<tr>
<td>Inverse transforms</td>
<td>130</td>
<td></td>
</tr>
<tr>
<td>The Dirac delta function – the unit impulse</td>
<td>134</td>
<td></td>
</tr>
<tr>
<td>Graphical representation</td>
<td>135</td>
<td></td>
</tr>
<tr>
<td>Laplace transform of $\delta(t - a)$</td>
<td>136</td>
<td></td>
</tr>
<tr>
<td>The derivative of the unit step function</td>
<td>139</td>
<td></td>
</tr>
</tbody>
</table>
## Programme 5  Difference equations and the Z transform

### Learning outcomes

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>155</td>
</tr>
<tr>
<td>Sequences</td>
<td>156</td>
</tr>
<tr>
<td>Difference equations</td>
<td>156</td>
</tr>
<tr>
<td>Solving difference equations</td>
<td>158</td>
</tr>
<tr>
<td>Solution by inspection</td>
<td>160</td>
</tr>
<tr>
<td>The particular solution</td>
<td>160</td>
</tr>
<tr>
<td>The Z transform</td>
<td>163</td>
</tr>
<tr>
<td>Table of Z transforms</td>
<td>166</td>
</tr>
<tr>
<td>Properties of Z transforms</td>
<td>171</td>
</tr>
<tr>
<td>Linearity</td>
<td>171</td>
</tr>
<tr>
<td>First shift theorem (shifting to the left)</td>
<td>172</td>
</tr>
<tr>
<td>Second shift theorem (shifting to the right)</td>
<td>173</td>
</tr>
<tr>
<td>Translation</td>
<td>174</td>
</tr>
<tr>
<td>Final value theorem</td>
<td>175</td>
</tr>
<tr>
<td>The initial value theorem</td>
<td>176</td>
</tr>
<tr>
<td>The derivative of the transform</td>
<td>176</td>
</tr>
<tr>
<td>Inverse transforms</td>
<td>177</td>
</tr>
<tr>
<td>Solving difference equations</td>
<td>180</td>
</tr>
<tr>
<td>Sampling</td>
<td>183</td>
</tr>
<tr>
<td>Revision summary 5</td>
<td>186</td>
</tr>
<tr>
<td>Can you? Checklist 5</td>
<td>189</td>
</tr>
<tr>
<td>Test exercise 5</td>
<td>190</td>
</tr>
<tr>
<td>Further problems 5</td>
<td>190</td>
</tr>
</tbody>
</table>

## Programme 6  Introduction to invariant linear systems

### Learning outcomes

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Invariant linear systems</td>
<td>193</td>
</tr>
<tr>
<td>Systems</td>
<td>194</td>
</tr>
<tr>
<td>Input-response relationships</td>
<td>194</td>
</tr>
<tr>
<td>Linear systems</td>
<td>195</td>
</tr>
<tr>
<td>Time-invariance of a continuous system</td>
<td>196</td>
</tr>
<tr>
<td>Shift-invariance of a discrete system</td>
<td>199</td>
</tr>
<tr>
<td>Differential equations</td>
<td>201</td>
</tr>
<tr>
<td>The general $n$th-order equation</td>
<td>202</td>
</tr>
</tbody>
</table>
Zero-input response and zero-state response 203
  Zero-input, zero-response 206
  Time-invariance 208
  Responses of a continuous system 209
    Impulse response 209
    Arbitrary input 209
    Exponential response 213
    The transfer function 215
    Differential equations 217
  Responses of a discrete system 220
    The discrete unit impulse 220
    Arbitrary input 221
    Exponential response 223
    Transfer function 224
    Difference equations 225
Revision summary 6 229
Can you? Checklist 6 232
Test exercise 6 233
Further problems 6 234
Programme 7  Fourier series 1 236

Learning outcomes 236
  Introduction 237
    Periodic functions 237
    Graphs of $y = A \sin nx$ 237
    Harmonics 238
    Non-sinusoidal periodic functions 239
    Analytic description of a periodic function 239
    Integrals of periodic functions 243
      Orthogonal functions 247
      Fourier series 247
      Dirichlet conditions 250
        Effect of harmonics 257
        Gibbs’ phenomenon 258
        Sum of a Fourier series at a point of discontinuity 259
Revision summary 7 261
Can you? Checklist 7 262
Test exercise 7 263
Further problems 7 264
Programme 8  Fourier series 2 267

Learning outcomes 267
  Functions with periods other than $2\pi$ 268
    Function with period $T$ 268
    Fourier coefficients 269
    Odd and even functions 272
    Products of odd and even functions 275
Half-range series 282
Series containing only odd harmonics or only even harmonics 286
Significance of the constant term $\frac{1}{2}a_0$ 288
Half-range series with arbitrary period 289

Revision summary 8
Can you? Checklist 8
Test exercise 8
Further problems 8

<table>
<thead>
<tr>
<th>Programme 9</th>
<th>Introduction to the Fourier transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>Learning outcomes 297</td>
<td></td>
</tr>
<tr>
<td>Complex Fourier series 298</td>
<td></td>
</tr>
<tr>
<td>Introduction 298</td>
<td></td>
</tr>
<tr>
<td>Complex exponentials 298</td>
<td></td>
</tr>
<tr>
<td>Complex spectra 303</td>
<td></td>
</tr>
<tr>
<td>The two domains 304</td>
<td></td>
</tr>
<tr>
<td>Continuous spectra 305</td>
<td></td>
</tr>
<tr>
<td>Fourier’s integral theorem 307</td>
<td></td>
</tr>
<tr>
<td>Some special functions and their transforms 310</td>
<td></td>
</tr>
<tr>
<td>Even functions 310</td>
<td></td>
</tr>
<tr>
<td>Odd functions 310</td>
<td></td>
</tr>
<tr>
<td>Top-hat function 312</td>
<td></td>
</tr>
<tr>
<td>The Dirac delta 314</td>
<td></td>
</tr>
<tr>
<td>The triangle function 316</td>
<td></td>
</tr>
<tr>
<td>Alternative forms 316</td>
<td></td>
</tr>
<tr>
<td>Properties of the Fourier transform 317</td>
<td></td>
</tr>
<tr>
<td>Linearity 317</td>
<td></td>
</tr>
<tr>
<td>Time shifting 318</td>
<td></td>
</tr>
<tr>
<td>Frequency shifting 318</td>
<td></td>
</tr>
<tr>
<td>Time scaling 318</td>
<td></td>
</tr>
<tr>
<td>Symmetry 319</td>
<td></td>
</tr>
<tr>
<td>Differentiation 320</td>
<td></td>
</tr>
<tr>
<td>The Heaviside unit step function 321</td>
<td></td>
</tr>
<tr>
<td>Convolution 322</td>
<td></td>
</tr>
<tr>
<td>The convolution theorem 323</td>
<td></td>
</tr>
<tr>
<td>Fourier cosine and sine transforms 325</td>
<td></td>
</tr>
<tr>
<td>Table of transforms 327</td>
<td></td>
</tr>
<tr>
<td>Revision summary 9</td>
<td></td>
</tr>
<tr>
<td>Can you? Checklist 9</td>
<td></td>
</tr>
<tr>
<td>Test exercise 9</td>
<td></td>
</tr>
<tr>
<td>Further problems 9</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Programme 10</th>
<th>Power series solutions of ordinary differential equations 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Learning outcomes 334</td>
<td></td>
</tr>
<tr>
<td>Higher derivatives 335</td>
<td></td>
</tr>
<tr>
<td>Leibnitz theorem 338</td>
<td></td>
</tr>
<tr>
<td>Programme 11</td>
<td>Power series solutions of ordinary differential equations 2</td>
</tr>
<tr>
<td>--------------</td>
<td>---------------------------------------------------------------</td>
</tr>
<tr>
<td>Learning outcomes</td>
<td>357</td>
</tr>
<tr>
<td>Introduction</td>
<td>358</td>
</tr>
<tr>
<td>Solution of differential equations by the method of Frobenius</td>
<td>358</td>
</tr>
<tr>
<td>Indicial equation</td>
<td>360</td>
</tr>
<tr>
<td>Revision summary 11</td>
<td>376</td>
</tr>
<tr>
<td>Can you? Checklist 11</td>
<td>377</td>
</tr>
<tr>
<td>Test exercise 11</td>
<td>377</td>
</tr>
<tr>
<td>Further problems 11</td>
<td>377</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Programme 12</th>
<th>Power series solutions of ordinary differential equations 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Learning outcomes</td>
<td>378</td>
</tr>
<tr>
<td>Introduction</td>
<td>379</td>
</tr>
<tr>
<td>Bessel functions</td>
<td>380</td>
</tr>
<tr>
<td>Graphs of Bessel functions $J_0(x)$ and $J_1(x)$</td>
<td>385</td>
</tr>
<tr>
<td>Legendre’s equation</td>
<td>385</td>
</tr>
<tr>
<td>Legendre polynomials</td>
<td>386</td>
</tr>
<tr>
<td>Rodrigue’s formula and the generating function</td>
<td>386</td>
</tr>
<tr>
<td>Sturm–Liouville systems</td>
<td>388</td>
</tr>
<tr>
<td>Orthogonality</td>
<td>390</td>
</tr>
<tr>
<td>Legendre’s equation revisited</td>
<td>391</td>
</tr>
<tr>
<td>Polynomials as a finite series of Legendre polynomials</td>
<td>392</td>
</tr>
<tr>
<td>Revision summary 12</td>
<td>393</td>
</tr>
<tr>
<td>Can you? Checklist 12</td>
<td>395</td>
</tr>
<tr>
<td>Test exercise 12</td>
<td>396</td>
</tr>
<tr>
<td>Further problems 12</td>
<td>396</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Programme 13</th>
<th>Numerical solutions of ordinary differential equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Learning outcomes</td>
<td>398</td>
</tr>
<tr>
<td>Introduction</td>
<td>399</td>
</tr>
<tr>
<td>Taylor’s series</td>
<td>399</td>
</tr>
<tr>
<td>Function increment</td>
<td>400</td>
</tr>
<tr>
<td>First-order differential equations</td>
<td>401</td>
</tr>
<tr>
<td>Euler’s method</td>
<td>401</td>
</tr>
</tbody>
</table>
The exact value and the errors  410
Graphical interpretation of Euler’s method  414
The Euler–Cauchy method – or the improved Euler method  416
Euler–Cauchy calculations  417
Runge–Kutta method  422
Second-order differential equations  425
Euler second-order method  425
Runge–Kutta method for second-order differential equations  427
Predictor–corrector methods  432
Revision summary  434
Can you? Checklist  436
Test exercise  436
Further problems  437

Programme 14 Partial differentiation  439

Learning outcomes  439
   Small increments  440
      Taylor’s theorem for one independent variable  440
      Taylor’s theorem for two independent variables  440
   Small increments  442
   Rates of change  444
   Implicit functions  445
   Change of variables  446
   Inverse functions  450
      General case  452
   Stationary values of a function  458
      Maximum and minimum values  459
      Saddle point  465
   Lagrange undetermined multipliers  470
      Functions with three independent variables  473
Revision summary  476
Can you? Checklist  478
Test exercise  479
Further problems  480

Programme 15 Partial differential equations  482

Learning outcomes  482
   Introduction  483
   Partial differential equations  484
      Solution by direct integration  484
      Initial conditions and boundary conditions  485
      The wave equation  486
      Solution of the wave equation  487
      Solution by separating the variables  487
   The heat conduction equation for a uniform finite bar  496
      Solutions of the heat conduction equation  497
Laplace's equation 502
Solution of the Laplace equation 502
Laplace's equation in plane polar coordinates 507
The problem 508
Separating the variables 508
The $n = 0$ case 511
Revision summary 15
Can you? Checklist 15
Test exercise 15
Further problems 15

Programme 16  Matrix algebra  519
Learning outcomes  519
Singular and non-singular matrices  520
Rank of a matrix  521
Elementary operations and equivalent matrices  522
Consistency of a set of equations  526
Uniqueness of solutions  527
Solution of sets of equations  531
Inverse method  531
Row transformation method  535
Gaussian elimination method  539
Triangular decomposition method  542
Using an electronic spreadsheet  548
Comparison of methods  552
Matrix transformation  553
Rotation of axes  555
Revision summary 16
Can you? Checklist 16
Test exercise 16
Further problems 16

Programme 17  Systems of ordinary differential equations  563
Learning outcomes  563
Eigenvalues and eigenvectors  564
Introduction  564
Cayley–Hamilton theorem  571
Systems of first-order ordinary differential equations  572
Diagonalisation of a matrix  577
Systems of second-order differential equations  582
Revision summary 17
Can you? Checklist 17
Test exercise 17
Further problems 17
Programme 18 Numerical solutions of partial differential equations

<table>
<thead>
<tr>
<th>Learning outcomes</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>593</td>
</tr>
<tr>
<td>Numerical...</td>
<td>594</td>
</tr>
<tr>
<td>Functions of two</td>
<td>597</td>
</tr>
<tr>
<td>Grid values</td>
<td>598</td>
</tr>
<tr>
<td>Computational...</td>
<td>601</td>
</tr>
<tr>
<td>Summary of...</td>
<td>605</td>
</tr>
<tr>
<td>Derivative...</td>
<td>608</td>
</tr>
<tr>
<td>Second-order...</td>
<td>612</td>
</tr>
<tr>
<td>Second partial...</td>
<td>615</td>
</tr>
<tr>
<td>Time-dependent</td>
<td>619</td>
</tr>
<tr>
<td>The Crank–Nicolson procedure</td>
<td>624</td>
</tr>
<tr>
<td>Dimensional...</td>
<td>631</td>
</tr>
</tbody>
</table>

Revision summary 18
Can you? Checklist 18
Test exercise 18
Further problems 18

Programme 19 Multiple integration 1

<table>
<thead>
<tr>
<th>Learning outcomes</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>642</td>
</tr>
<tr>
<td>Differentials</td>
<td>650</td>
</tr>
<tr>
<td>Exact differential</td>
<td>653</td>
</tr>
<tr>
<td>Integration of...</td>
<td>655</td>
</tr>
<tr>
<td>Area enclosed by</td>
<td>657</td>
</tr>
<tr>
<td>Line integrals</td>
<td>660</td>
</tr>
<tr>
<td>Alternative form</td>
<td>661</td>
</tr>
<tr>
<td>Properties of...</td>
<td>664</td>
</tr>
<tr>
<td>Regions...</td>
<td>666</td>
</tr>
<tr>
<td>Line integrals...</td>
<td>667</td>
</tr>
<tr>
<td>Line integral...</td>
<td>671</td>
</tr>
<tr>
<td>Parametric...</td>
<td>672</td>
</tr>
<tr>
<td>Dependence of...</td>
<td>673</td>
</tr>
<tr>
<td>Exact differentials in three independent variables</td>
<td>678</td>
</tr>
<tr>
<td>Green’s theorem</td>
<td>679</td>
</tr>
</tbody>
</table>

Revision summary 19
Can you? Checklist 19
Test exercise 19
Further problems 19

Programme 20 Multiple integration 2

<table>
<thead>
<tr>
<th>Learning outcomes</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Double integrals</td>
<td>691</td>
</tr>
<tr>
<td>Surface integrals</td>
<td>697</td>
</tr>
</tbody>
</table>
Programme 21 Integral functions

Learning outcomes

Integral functions

The gamma function

The beta function

Relation between the gamma and beta functions

Application of gamma and beta functions

Duplication formula for gamma functions

The error function

The graph of erf (x)

The complementary error function erfc (x)

Elliptic functions

Standard forms of elliptic functions

Complete elliptic functions

Alternative forms of elliptic functions

Revision summary 21

Can you? Checklist 21

Test exercise 21

Further problems 21

Programme 22 Vector analysis 1

Learning outcomes

Introduction

Triple products

Properties of scalar triple products

Coplanar vectors

Vector triple products of three vectors

Differentiation of vectors

Differentiation of sums and products of vectors

Unit tangent vectors

Partial differentiation of vectors

Integration of vector functions

Scalar and vector fields

Grad (gradient of a scalar field)

Directional derivatives

Unit normal vectors
<table>
<thead>
<tr>
<th>Castrol's of sums and products of scalars</th>
<th>803</th>
</tr>
</thead>
<tbody>
<tr>
<td>Div (divergence of a vector function)</td>
<td>805</td>
</tr>
<tr>
<td>Curl (curl of a vector function)</td>
<td>806</td>
</tr>
<tr>
<td>Summary of grad, div and curl</td>
<td>807</td>
</tr>
<tr>
<td>Multiple operations</td>
<td>809</td>
</tr>
<tr>
<td>Revision summary 22</td>
<td>812</td>
</tr>
<tr>
<td>Can you? Checklist 22</td>
<td>814</td>
</tr>
<tr>
<td>Test exercise 22</td>
<td>815</td>
</tr>
<tr>
<td>Further problems 22</td>
<td>815</td>
</tr>
</tbody>
</table>

**Programme 23  Vector analysis 2  818**

<table>
<thead>
<tr>
<th>Learning outcomes</th>
<th>818</th>
</tr>
</thead>
<tbody>
<tr>
<td>Line integrals</td>
<td>819</td>
</tr>
<tr>
<td>Scalar field</td>
<td>819</td>
</tr>
<tr>
<td>Vector field</td>
<td>822</td>
</tr>
<tr>
<td>Volume integrals</td>
<td>826</td>
</tr>
<tr>
<td>Surface integrals</td>
<td>830</td>
</tr>
<tr>
<td>Scalar fields</td>
<td>831</td>
</tr>
<tr>
<td>Vector fields</td>
<td>834</td>
</tr>
<tr>
<td>Conservative vector fields</td>
<td>839</td>
</tr>
<tr>
<td>Divergence theorem (Gauss’ theorem)</td>
<td>844</td>
</tr>
<tr>
<td>Stokes’ theorem</td>
<td>850</td>
</tr>
<tr>
<td>Direction of unit normal vectors to a surface S</td>
<td>853</td>
</tr>
<tr>
<td>Green’s theorem</td>
<td>859</td>
</tr>
<tr>
<td>Revision summary 23</td>
<td>862</td>
</tr>
<tr>
<td>Can you? Checklist 23</td>
<td>864</td>
</tr>
<tr>
<td>Test exercise 23</td>
<td>865</td>
</tr>
<tr>
<td>Further problems 23</td>
<td>866</td>
</tr>
</tbody>
</table>

**Programme 24  Vector analysis 3  869**

<table>
<thead>
<tr>
<th>Learning outcomes</th>
<th>869</th>
</tr>
</thead>
<tbody>
<tr>
<td>Curvilinear coordinates</td>
<td>870</td>
</tr>
<tr>
<td>Orthogonal curvilinear coordinates</td>
<td>874</td>
</tr>
<tr>
<td>Orthogonal coordinate systems in space</td>
<td>875</td>
</tr>
<tr>
<td>Scale factors</td>
<td>879</td>
</tr>
<tr>
<td>Scale factors for coordinate systems</td>
<td>880</td>
</tr>
<tr>
<td>General curvilinear coordinate system (u, v, w)</td>
<td>882</td>
</tr>
<tr>
<td>Transformation equations</td>
<td>883</td>
</tr>
<tr>
<td>Element of arc ds and element of volume dV in orthogonal curvilinear coordinates</td>
<td>884</td>
</tr>
<tr>
<td>Grad, div and curl in orthogonal curvilinear coordinates</td>
<td>885</td>
</tr>
<tr>
<td>Particular orthogonal systems</td>
<td>888</td>
</tr>
<tr>
<td>Revision summary 24</td>
<td>890</td>
</tr>
<tr>
<td>Can you? Checklist 24</td>
<td>892</td>
</tr>
<tr>
<td>Test exercise 24</td>
<td>893</td>
</tr>
<tr>
<td>Further problems 24</td>
<td>894</td>
</tr>
</tbody>
</table>
Programme 25  Complex analysis 1

Learning outcomes
- Functions of a complex variable
- Complex mapping
  - Mapping of a straight line in the z-plane onto the w-plane under the transformation $w = f(z)$
  - Types of transformation of the form $w = az + b$
- Non-linear transformations
  - Mapping of regions
Revision summary 25
Can you? Checklist 25
Test exercise 25
Further problems 25

Programme 26  Complex analysis 2

Learning outcomes
- Differentiation of a complex function
  - Regular function
  - Cauchy–Riemann equations
- Harmonic functions
- Complex integration
  - Contour integration – line integrals in the z-plane
  - Cauchy's theorem
  - Deformation of contours at singularities
- Conformal transformation (conformal mapping)
  - Conditions for conformal transformation
  - Critical points
  - Schwarz–Christoffel transformation
  - Open polygons
Revision summary 26
Can you? Checklist 26
Test exercise 26
Further problems 26

Programme 27  Complex analysis 3

Learning outcomes
- Maclaurin series
- Radius of convergence
- Singular points
  - Poles
  - Removable singularities
- Circle of convergence
- Taylor’s series
- Laurent’s series
Programme 28  Optimization and linear programming  1014

Learning outcomes 1014
  Optimization 1015
    Linear programming (or linear optimization) 1015
    Linear inequalities 1016
    Graphical representation of linear inequalities 1016
  The simplex method 1022
    Setting up the simplex tableau 1022
    Computation of the simplex 1024
    Simplex with three problem variables 1032
    Artificial variables 1036
    Minimisation 1047
  Applications 1051
Revision summary 28  1055
Can you? Checklist 28  1056
Test exercise 28  1057
Further problems 28  1058

Appendix  1063
Answers  1072
Index  1105
Programme 1

Numerical solutions of equations and interpolation

Learning outcomes

When you have completed this Programme you will be able to:

- Appreciate the Fundamental Theorem of Algebra
- Find the two roots of a quadratic equation and recognise that for polynomial equations with real coefficients complex roots exist in complex conjugate pairs
- Use the relationships between the coefficients and the roots of a polynomial equation to find the roots of the polynomial
- Transform a cubic equation to its reduced form
- Use Tartaglia’s solution to find the roots of a cubic equation
- Find the solution of the equation $f(x) = 0$ by the method of bisection
- Solve equations involving a single real variable by iteration and use a spreadsheet for efficiency
- Solve equations using the Newton–Raphson iterative method
- Use the modified Newton–Raphson method to find the first approximation when the derivative is small
- Understand the meaning of interpolation and use simple linear and graphical interpolation
- Use the Gregory–Newton interpolation formula with forward and backward differences for equally spaced domain points
- Use the Gauss interpolation formulas using central differences for equally spaced domain points
- Use Lagrange interpolation when the domain points are not equally spaced
Introduction

In this Programme we shall be looking at analytic and numerical methods of solving the general equation in a single variable, \( f(x) = 0 \). In addition, a functional relationship can be exhibited in the form of a collection of ordered pairs rather than in the form of an algebraic expression. We shall be looking at interpolation methods of estimating values of \( f(x) \) for intermediate values of \( x \) between those listed among the ordered pairs.

First we shall look at the **Fundamental Theorem of Algebra**, which deals with the factorisation of polynomials.

The Fundamental Theorem of Algebra

The *Fundamental Theorem of Algebra* can be stated as follows:

Every polynomial expression \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \) can be written as a product of \( n \) linear factors in the form

\[
 f(x) = a_n(x - r_1)(x - r_2)(\cdots)(x - r_n)
\]

As an immediate consequence of this we can see that there are \( n \) values of \( x \) that satisfy the polynomial equation \( f(x) = 0 \), namely \( x = r_1, x = r_2, \ldots, x = r_n \). We call these values the *roots* of the polynomial, but be aware that they may not all be distinct. Furthermore, the polynomial coefficients \( a_i \) and the polynomial roots \( r_i \) may be real, imaginary or complex.

For example the quadratic equation

\[
 x^2 + 5x + 6 = 0
\]

can be written \((x + 2)(x + 3) = 0\) so it has the two *distinct* roots \( x = -2 \) and \( x = -3 \).

\[
 x^2 - 4x + 4 = 0
\]

can be written as \((x - 2)(x - 2) = 0\) so it has the two *coincident* roots \( x = 2 \) and \( x = 2 \).

\[
 x^2 + x + 1 = 0
\]

can be written as \((x + a)(x + b) = 0\) so it has the two roots \( x = -a \) and \( x = -b \).

To find the numerical values of \( a \) and \( b \) we need to use the formula for finding the roots of a general quadratic equation. Can you recall what it is? If not, then refer to Frame 14 of Programme F.6 in *Engineering Mathematics, Sixth Edition*.

The solution to the quadratic equation \( ax^2 + bx + c = 0 \) is ............

*The answer is in the next frame*
So the roots of \( x^2 + x + 1 = 0 \) are .......... 

Because

\[
 a = b = c = 1 \quad \text{and so} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4}}{2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}
\]

This quadratic equation has two distinct complex roots. Notice that the two roots form a complex conjugate pair – each is the complex conjugate of the other. **Whenever a polynomial with real coefficients \( a_i \) has a complex root it also has the complex conjugate as another root.**

So given that \( x = -2 + j\sqrt{5} \) is one root of a quadratic equation with real coefficients then

the other root is .......... 

Because

The complex conjugate of \( x = -2 + j\sqrt{5} \) is \( x = -2 - j\sqrt{5} \) and complex roots of a polynomial equation with real coefficients always appear as conjugate pairs.

The quadratic equation with these two roots is ..........
Because

If \( x = a \) and \( x = b \) are the two roots of a quadratic equation then \((x - a)(x - b) = 0\) gives the quadratic equation. That is \((x - a)(x - b) = x^2 - (a + b)x + ab = 0\).

Here, the two roots are \( x = -2 + j\sqrt{5} \) and \( x = -2 - j\sqrt{5} \) so that

\[
\left(x - \left[-2 + j\sqrt{5}\right]\right)\left(x - \left[-2 - j\sqrt{5}\right]\right) = 0
\]

That is \( x^2 - x \left[-2 + j\sqrt{5} - 2 - j\sqrt{5}\right] + \left[-2 + j\sqrt{5}\right] \left[-2 - j\sqrt{5}\right] = 0. \)

So \( x^2 + 4x + 9 = 0. \)

Notice that the coefficients are \(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.
This, of course, applies to a cubic equation. Let us extend this to a more general equation.

In general, if \( \alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n \) are roots of the equation
\[
p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0 \quad (p_0 \neq 0)
\]
then
- sum of the roots \( = \frac{-p_1}{p_0} \)
- sum of products of the roots, two at a time \( = \frac{p_2}{p_0} \)
- sum of products of the roots, three at a time \( = -\frac{p_2}{p_0} \)
- sum of products of the roots, \( n \) at a time \( = (-1)^n \frac{p_n}{p_0} \)

So for the equation \( 3x^4 + 2x^3 + 5x^2 + 7x - 4 = 0 \), if \( \alpha, \beta, \gamma, \delta \) are the four roots, then
(a) \( \alpha + \beta + \gamma + \delta = \ldots \)
(b) \( \alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \delta\beta + \gamma\alpha = \ldots \)
(c) \( \alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \alpha\beta\delta = \ldots \)
(d) \( \alpha\beta\gamma\delta = \ldots \)

Now for a problem or two on the same topic.

**Example 1**

Solve the equation \( x^3 - 8x^2 + 9x + 18 = 0 \) given that the sum of two of the roots is 5.

Using the same approach as before, if \( \alpha, \beta, \gamma \) are the roots, then
(a) \( \alpha + \beta + \gamma = \ldots \)
(b) \( \alpha\beta + \beta\gamma + \gamma\alpha = \ldots \)
(c) \( \alpha\beta\gamma = \ldots \)
12

(a) 8; (b) 9; (c) –18

So we have \( \alpha + \beta + \gamma = 8 \)  
Let \( \alpha + \beta = 5 \)
\[ \therefore 5 + \gamma = 8 \quad \therefore \gamma = 3 \]
Also \( \alpha \beta \gamma = -18 \)  
\( \alpha \beta (3) = -18 \)  
\[ \therefore \alpha \beta = -6 \]
\( \alpha + \beta = 5 \)  
\[ \therefore \beta = 5 - \alpha \quad \therefore \alpha(5 - \alpha) = -6 \]
\( \alpha^2 - 5\alpha - 6 = 0 \)  
\[ \therefore (\alpha - 6)(\alpha + 1) = 0 \quad \therefore \alpha = -1 \text{ or } 6 \]
\[ \therefore \beta = 6 \text{ or } -1 \]

Roots are \( x = -1, 3, 6 \)

13

Example 2

Solve the equation \( 2x^3 + 3x^2 - 11x - 6 = 0 \) given that the three roots form an arithmetic sequence.

Let us represent the roots by \( (a - k), a, (a + k) \)

Then the sum of the roots = \( 3a = \ldots \ldots \ldots \)

and the product of the roots = \( a(a - k)(a + k) = \ldots \ldots \ldots \)

14

\[
3a = -\frac{3}{2}; \quad a(a + k)(a - k) = \frac{6}{2} = 3
\]

\[ \therefore a = -\frac{1}{2} - \frac{1}{2} \left(\frac{1}{4} - k^2\right) = 3 \quad \therefore k = \pm \frac{5}{2} \]

If \( k = \frac{5}{2} \)  
\[ a = -\frac{1}{2}; \quad a - k = -3; \quad a + k = 2 \]

If \( k = -\frac{5}{2} \)  
\[ a = -\frac{1}{2}; \quad a - k = 2; \quad a + k = -3 \]

\[ \therefore \text{required roots are } -3, -\frac{1}{2}, 2 \]

Here is a similar one.

Example 3

Solve the equation \( x^3 + 3x^2 - 6x - 8 = 0 \) given that the three roots are in geometric sequence.

This time, let the roots be \( \frac{a}{k}, a, ak \)

Then \( \frac{a}{k} = a + ak = \ldots \ldots \ldots \) and \( \left(\frac{a}{k}\right)(a)(ak) = \ldots \ldots \ldots \)
The working rests on the relationships between the roots and the coefficients, i.e. if $\alpha$, $\beta$, $\gamma$ are the roots of the cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

then

(a) $\alpha + \beta + \gamma = \ldots$ \\
(b) $\alpha\beta + \beta\gamma + \gamma\alpha = \ldots$ \\
(c) $\alpha\beta\gamma = \ldots$

In each of the three examples reconstruct the cubic to confirm that they are correct.

Now on to the next stage
To find the value of this real root we can employ a formula equivalent to the formula used to find the two roots of the general quadratic. This is called Tartaglia’s method but before we can proceed to look at that we must first consider how to transform the general cubic to its reduced form.

Transforming a cubic to reduced form

In every case, an equation of the form

\[ x^3 + ax^2 + bx + c = 0 \]

can be converted into the reduced form \( y^3 + py + q = 0 \) by the substitution \( x = y - \frac{a}{3} \).

The example will demonstrate the method.

Example 4

Express \( f(x) = x^3 + 6x^2 - 4x + 5 = 0 \) in reduced form.

Substitute \( x = y - \frac{a}{3} \) i.e. \( x = y - \frac{6}{3} = y - 2 \). Put \( x = y - 2 \).

The equation then becomes

\[ (y - 2)^3 + 6(y - 2)^2 - 4(y - 2) + 5 = 0 \]
\[ (y^3 - 3y^2 2 + 3y4 - 8) + 6(y^2 - 4y + 4) - 4(y - 2) + 5 = 0 \]

which simplifies to ............

Tartaglia’s solution for a real root

In the sixteenth century, Tartaglia discovered that a root of the cubic equation \( x^3 + ax + b = 0 \), where \( a > 0 \), is given by

\[
x = \left\{ \frac{b}{2} + \sqrt{\frac{a^3}{27} + \frac{b^2}{4}} \right\}^{1/3} + \left\{ -\frac{b}{2} - \sqrt{\frac{a^3}{27} + \frac{b^2}{4}} \right\}^{1/3}
\]

That looks pretty formidable, but it is a good deal easier than it appears. Notice that \( \frac{b}{2} \) and \( \sqrt{\frac{a^3}{27} + \frac{b^2}{4}} \) occur twice and it is convenient to evaluate these first and then substitute the results in the main expression for \( x \).
Example 5

Find a real root of \(x^3 + 2x + 5 = 0\).

Here, \(a = 2, \ b = 5\) \(\therefore \ \frac{b}{2} = 2.5\)

\[
\sqrt{\frac{a^3}{27} + \frac{b^2}{4}} = \sqrt{\frac{8}{27} + \frac{25}{4}} = \sqrt{6.5463} = 2.5586
\]

Then \(x = (-2.5 + 2.5586)^{1/3} + (-2.5 - 2.5586)^{1/3}\)

\[
= 0.3884 - 1.7166 = -1.3282 \quad x = -1.328
\]

Once we have a real root, the equation can be reduced to a quadratic and the remaining two roots determined. They are \(x = 0.664 + j1.823\) and \(x = 0.664 - j1.823\) (see Engineering Mathematics, Sixth Edition, Programme F.6).

Example 6

Determine a real root of \(2x^3 + 3x - 4 = 0\).

This is first written \(x^3 + 1.5x - 2 = 0\) \(\therefore \ a = 1.5, \ b = -2\)

Now you can evaluate \(\frac{b}{2}\) and \(\sqrt{\frac{a^3}{27} + \frac{b^2}{4}}\) and so determine

\[
x = \ldots \ldots \ldots
\]

\[0.8796\]

Because

\[
\left\{-\frac{b}{2} + \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}\right\}^{1/3} = (2.06066)^{1/3} = 1.2725 \text{ and }
\]

\[
\left\{-\frac{b}{2} - \sqrt{\frac{a^3}{27} + \frac{b^2}{4}}\right\}^{1/3} = (-0.6066)^{1/3} = -0.3929,
\]

therefore \(x = 1.2725 - 0.3929 = 0.8796\)

Note: If you wish to find the real root of a cubic of the form \(x^3 + ax + b = 0\) where \(a < 0\) then it is best that you resort to numerical methods. Read on.
Numerical methods

The methods that we have used so far to solve quadratic equations and to find the real root of a cubic equation are called analytic methods. These analytic methods used straightforward algebraic techniques to develop a formula for the answer. The numerical value of the answer can then be found by simple substitution of numbers for the variables in the formula. Unfortunately, general polynomial equations of order five or higher cannot by solved by analytic methods. Instead, we must resort to what are termed numerical methods. The simplest method of finding the solution to the equation \( f(x) = 0 \) is the bisection method.

**Bisection**

The bisection method of finding a solution to the equation \( f(x) = 0 \) consists of

1. Finding a value of \( x \), say \( x = a \), such that \( f(a) < 0 \)
2. Finding a value of \( x \), say \( x = b \), such that \( f(b) > 0 \)

The solution to the equation \( f(x) = 0 \) must then lie between \( a \) and \( b \). Furthermore, it must lie either in the first half of the interval between \( a \) and \( b \) or in the second half.

![Bisection Method Diagram](image)

Find the value of \( f((a + b)/2) \) – that is halfway between \( a \) and \( b \).

If \( f((a + b)/2) > 0 \) then the solution lies in the first half and if \( f((a + b)/2) < 0 \) then it lies in the second half. This procedure is repeated, narrowing down the width of the interval by a half each time. An example should clarify all this.

**Example 7**

Find the positive value of \( x \) that satisfies the equation \( x^2 - 2 = 0 \).

Firstly we note that if \( x = 1 \) then \( x^2 - 2 < 0 \), and that if \( x = 2 \) then \( x^2 - 2 > 0 \), so the solution that we seek must lie between 1 and 2.

We look for the ............
The mid-point between 1 and 2 which is 1.5

Now, when \( x = 1.5 \), \( x^2 - 2 = 0.25 > 0 \)
so the solution must lie between ...........

1 and 1.5

The mid-point between 1 and 1.5 is 1.25. When \( x = 1.25 \), \( x^2 - 2 = -0.4375 < 0 \)
so the solution must lie between ...........

1.25 and 1.5

The mid-point between 1.25 and 1.5 is 1.375. We now evaluate \( x^2 - 2 \) at this point and determine in which half interval the solution lies. This process is repeated and the following table displays the results. In each block of six numbers the first column lists the end points of the interval and the mid-point. The second column contains the respective values \( f(x) = x^2 - 2 \). Construct the table as follows.

(a) For each block of six numbers copy the last number in the first column into the second place of the first column of the following block. This represents the centre point of the previous interval.

(b) For each block of six numbers copy the number that represents the other end point of the new interval from the first column into the first place of the first column of the following block. Look at the signs in the second column to decide which is the appropriate number.

\[
\begin{array}{cccccc}
a & 1.0000 & -1.0000 & 1.0000 & -1.0000 & 1.5000 & 0.2500 \\
b & 2.0000 & 2.0000 & 1.5000 & -0.2500 & 1.2500 & 0.4375 \\
(a + b)/2 & 1.5000 & -0.2500 & 1.2500 & -0.4375 & 1.3750 & -0.1094 \\
\end{array}
\]

\[
\begin{array}{cccccc}
a & 1.3750 & -0.1094 & 1.4375 & 0.0664 & 1.4063 & -0.0225 \\
b & 1.4375 & 0.0664 & 1.4063 & -0.0225 & 1.4219 & 0.0217 \\
(a + b)/2 & 1.4063 & -0.0225 & 1.4219 & 0.0217 & 1.4141 & -0.0004 \\
\end{array}
\]

\[
\begin{array}{cccccc}
a & 1.4141 & -0.0004 & 1.4141 & -0.0004 & 1.4141 & -0.0004 \\
b & 1.4180 & 0.0106 & 1.4160 & 0.0051 & 1.4150 & 0.0023 \\
(a + b)/2 & 1.4160 & 0.0051 & 1.4150 & 0.0023 & 1.4146 & 0.0010 \\
\end{array}
\]

\[
\begin{array}{cccccc}
a & 1.4141 & -0.0004 & 1.4143 & 0.0003 & 1.4142 & -0.0001 \\
b & 1.4143 & 0.0003 & 1.4142 & -0.0001 & 1.4142 & 0.0001 \\
(a + b)/2 & 1.4142 & -0.0001 & 1.4142 & 0.0001 & 1.4142 & 0.0000 \\
\end{array}
\]

The final result to four decimal places is \( x = 1.4142 \) which is the correct answer to that level of accuracy – but it has taken a lot of activity to produce it. A much faster way of solving this equation is to use an iteration formula that was first devised by Newton.
Numerical solution of equations by iteration

The process of finding the numerical solution to the equation
\[ f(x) = 0 \]
by iteration is performed by first finding an approximate solution and then using this approximate solution to find a more accurate solution. This process is repeated until a solution is found to the required level of accuracy. For example, Newton showed that the square root of a number \( a \) can be found from the iteration equation
\[ x_{i+1} = \frac{1}{2} \left( x_i + \frac{a}{x_i} \right), \quad i = 0, 1, 2, \ldots \]
where \( x_0 \) is the approximation that starts the iteration off. So, to find a succession of approximate values of \( \sqrt{2} \), each of increasing accuracy, we proceed as follows. Let \( x_0 = 1.5 \) – found by the first stage of the bisection method. Then
\[ x_1 = \frac{1}{2} \left( x_0 + \frac{a}{x_0} \right) = 0.5(1.5 + 2/1.5) = 1.4166 \ldots \]
This value is then used to find \( x_2 \).

By rounding \( x_1 \) to 1.4167, the value of \( x_2 \) is found to be \ldots \ldots

\[ x_2 = 1.4142 \]

Because
\[ x_2 = \frac{1}{2} \left( x_1 + \frac{a}{x_1} \right) = 0.5(1.4167 + 2/1.4167) = 1.4142 \ldots \]
This has achieved the same level of accuracy as the bisection method in just two steps.

Using a spreadsheet

This simple iteration procedure is more efficiently performed using a spreadsheet. If the use of a spreadsheet is a totally new experience for you then you are referred to Programme 4 of Engineering Mathematics, Sixth Edition where the spreadsheet is introduced as a tool for constructing graphs of functions. If you have a limited knowledge then you will be able to follow the text from here. The spreadsheet we shall be using here is Microsoft Excel, though all commercial spreadsheets possess the equivalent functionality.

Open your spreadsheet and in cell A1 enter \( n \) and press Enter. In this first column we are going to enter the iteration numbers. In cell A2 enter the number 0 and press Enter. Place the cell highlight in cell A2 and highlight the block of cells A2 to A7 by holding down the mouse button and wiping the highlight down to cell A7. Click the Edit command on the Command bar and
point at Fill from the drop-down menu. Select Series from the next drop-
down menu and accept the default Step value of 1 by clicking OK in the Series
window.

The cells A3 to A7 fill with ...........

<table>
<thead>
<tr>
<th>n</th>
<th>x</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.5</td>
</tr>
<tr>
<td>1</td>
<td>1.416667</td>
</tr>
<tr>
<td>2</td>
<td>1.414216</td>
</tr>
<tr>
<td>3</td>
<td>1.414214</td>
</tr>
<tr>
<td>4</td>
<td>1.414214</td>
</tr>
<tr>
<td>5</td>
<td>1.414214</td>
</tr>
</tbody>
</table>

In cell B1 enter the letter x – this column is going to contain the successive x-
values obtained by iteration. In cell B2 enter the value of $x_0$, namely 1.5.
In cell B3 enter the formula

$$= 0.5*(B2+2/B2)$$

The number that appears in cell B3 is then ............

1.416667

Place the cell highlight in cell B3, click the command Edit on the Command
bar and select Copy from the drop-down menu. You have now copied the
formula in cell B3 onto the Clipboard. Highlight the cells B4 to B7 and then
click the Edit command again but this time select Paste from the drop-down
menu.

The cells B4 to B7 fill with numbers to provide the display

............

By using the various formatting facilities provided by the spreadsheet the
display can be amended to provide the following

<table>
<thead>
<tr>
<th>n</th>
<th>x</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.5000000000000000</td>
</tr>
<tr>
<td>1</td>
<td>1.4166666666666670</td>
</tr>
<tr>
<td>2</td>
<td>1.414215686274510</td>
</tr>
<tr>
<td>3</td>
<td>1.414213562374690</td>
</tr>
<tr>
<td>4</td>
<td>1.414213562373090</td>
</tr>
<tr>
<td>5</td>
<td>1.414213562373090</td>
</tr>
</tbody>
</table>

The number of decimal places here is 15, which is far greater than is normally
required but it does demonstrate how effective a spreadsheet can be. In future
we shall restrict the displays to 6 decimal places.
Notice that to find a value accurate to a given number of decimal places or significant figures it is sufficient to repeat the iterations until there is no change in the result from one iteration to the next.

Save your spreadsheet under some suitable name such as Newton because you may wish to use it again.

Now we shall look at this spreadsheet a little more closely

### 32 Relative addresses

Place the cell highlight in cell B3 and the formula that it contains is $= 0.5 \times (B2 + 2/B2)$. Now place the cell highlight in cell B4 and the formula there is $= 0.5 \times (B3 + 2/B3)$. Why the difference?

When you enter the cell address B2 in the formula in B3 the spreadsheet understands that to mean the contents of the cell immediately above. It is this meaning that is copied into cell B4 where the cell immediately above is B3. If you wish to refer to a specific cell in a formula then you must use an absolute address.

Place the cell highlight in cell C1 and enter the number 2. Now place the cell highlight in cell B3 and re-enter the formula

$$= 0.5 \times (B2 + $C$1/B2)$$

and copy this into cells B4 to B7. The numbers in the second column have not changed but the formulas have because in cells B3 to B7 the same reference is made to cell C1. The use of the dollar signs has indicated an absolute address. So why would we do this?

Change the number in cell C1 to 3 to obtain the display ............

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.500000000000000000000</td>
</tr>
<tr>
<td>1</td>
<td>1.750000000000000000000</td>
</tr>
<tr>
<td>2</td>
<td>1.732142857142860000000</td>
</tr>
<tr>
<td>3</td>
<td>1.732050810014730000000</td>
</tr>
<tr>
<td>4</td>
<td>1.732050807568880000000</td>
</tr>
<tr>
<td>5</td>
<td>1.732050807568880000000</td>
</tr>
</tbody>
</table>

These are the iterated values of $\sqrt{3}$ – the square root of the contents of cell C1. We can now use the same spreadsheet to find the square root of any positive number.

Newton’s iterative procedure to find the square root of a positive number is a special case of the Newton–Raphson procedure to find the solution of the general equation $f(x) = 0$, and we shall look at this in the next frame.
Newton–Raphson iterative method

Consider the graph of \( y = f(x) \) as shown. Then the \( x \)-value at the point A, where the graph crosses the \( x \)-axis, gives a solution of the equation \( f(x) = 0 \).

If \( P \) is a point on the curve near to A, then \( x = x_0 \) is an approximate value of the root of \( f(x) = 0 \), the error of the approximation being given by AB.

Let PQ be the tangent to the curve as P, crossing the \( x \)-axis at Q \((x_1, 0)\). Then \( x = x_1 \) is a better approximation to the required root.

From the diagram, \( \frac{PB}{QB} = \left[ \frac{dy}{dx} \right]_P \) i.e. the value of the derivative of \( y \) at the point \( P \), \( x = x_0 \).

\[
\begin{align*}
\therefore \quad \frac{PB}{QB} &= f'(x_0) \quad \text{and} \quad PB = f(x_0) \\
\therefore \quad QB &= \frac{PB}{f'(x_0)} = f(x_0) f'(x_0) = h \quad \text{(say)} \\
x_1 &= x_0 - h \quad \therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}
\end{align*}
\]

If we begin, therefore, with an approximate value \( (x_0) \) of the root, we can determine a better approximation \( (x_1) \). Naturally, the process can be repeated to improve the result still further. Let us see this in operation.

**Example 1**

The equation \( x^3 - 3x - 4 = 0 \) is of the form \( f(x) = 0 \) where \( f(1) < 0 \) and \( f(3) > 0 \) so there is a solution to the equation between 1 and 3. We shall take this to be 2, by bisection. Find a better approximation to the root.

We have \( f(x) = x^3 - 3x - 4 \quad \therefore \quad f'(x) = 3x^2 - 3 \)

If the first approximation is \( x_0 = 2 \), then

\[
\begin{align*}
f(x_0) &= f(2) = -2 \quad \text{and} \quad f'(x_0) = f'(2) = 9
\end{align*}
\]

A better approximation \( x_1 \) is given by

\[
x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^3 - 3x_0 - 4}{3x_0^2 - 3}
\]

\[
x_1 = 2 - \frac{(-2)}{9} = 2.22
\]

\[
\therefore \quad x_0 = 2; \quad x_1 = 2.22
\]
If we now start from \( x_1 \) we can get a better approximation still by repeating the process.

\[
x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1^3 - 3x_1 - 4}{3x_1^2 - 3}
\]

Here \( x_1 = 2.22 \)

\( f(x_1) = \ldots \ldots \) \( f'(x_1) = \ldots \ldots \)

36

\[
f(x_1) = 0.281; \quad f'(x_1) = 11.785
\]

Then \( x_2 = \ldots \ldots \)

37

\[
x_2 = 2.196
\]

Because

\[
x_2 = 2.22 - \frac{0.281}{11.79} = 2.196
\]

Using \( x_2 = 2.196 \) as a starter value, we can continue the process until successive results agree to the desired degree of accuracy.

\( x_3 = \ldots \ldots \)

38

\[
x_3 = 2.196
\]

Because

\[
f(x_2) = f(2.196) = 0.002026; \quad f'(x_2) = f'(2.196) = 11.467
\]

\[ . \quad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.196 - \frac{0.00203}{11.467} = 2.196 \) (to 4 sig. fig.)

The process is simple but effective and can be repeated again and again. Each repetition, or iteration, usually gives a result nearer to the required root \( x = x_A \).

In general \( x_{n+1} = \ldots \ldots \)

39

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]

**Tabular display of results**

Open your spreadsheet and in cells A1 to D1 enter the headings \( n \), \( x \), \( f(x) \) and \( f'(x) \).

Fill cells A2 to A6 with the numbers 0 to 4.

In cell B2 enter the value for \( x_0 \), namely 2.

In cell C2 enter the formula for \( f(x_0) \), namely \( B2^3 - 3*B2 - 4 \) and copy into cells C3 to C6.
In cell D2 enter the formula for $f'(x_0)$, namely $3B2^2 – 3$ and copy into cells D3 to D6.

In cell B3 enter the formula for $x_1$, namely $B2 – C2/D2$ and copy into cells B4 to B6.

The final display is ............

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x$</th>
<th>$f(x)$</th>
<th>$f'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>-2</td>
<td>9</td>
</tr>
<tr>
<td>1</td>
<td>2.222222</td>
<td>0.30727</td>
<td>11.81481</td>
</tr>
<tr>
<td>2</td>
<td>2.196215</td>
<td>0.004492</td>
<td>11.47008</td>
</tr>
<tr>
<td>3</td>
<td>2.195823</td>
<td>1.01E-06</td>
<td>11.46492</td>
</tr>
<tr>
<td>4</td>
<td>2.195823</td>
<td>5.15E-14</td>
<td>11.46492</td>
</tr>
</tbody>
</table>

As soon as the number in the second column is repeated then we know that we have arrived at that particular level of accuracy. The required root is therefore $x = 2.195823$ to 6 dp. Save the spreadsheet so that it can be used as a template for other such problems.

Now let us have another example.

Next frame

Example 2

The equation $x^3 + 2x^2 – 5x – 1 = 0$ is of the form $f(x) = 0$ where $f(1) < 0$ and $f(2) > 0$ so there is a solution to the equation between 1 and 2. We shall take this to be $x = 1.5$. Use the Newton–Raphson method to find the root to six decimal places.

Use the previous spreadsheet as a template and make the following amendments.

In cell B2 enter the number ............

Because

That is the value of $x_0$ that is used to start the iteration

In cell C2 enter the formula ............

Because

That is the value of $f(x_0) = x_0^3 + 2x_0^2 – 5x_0 – 1$. Copy the contents of cell C2 into cells C3 to C5.

In cell D2 enter the formula ............
Because

That is the value of \( f'(x_0) = 3x_0^2 + 4x_0 - 5 \). Copy the contents of cell D2 into cells D3 to D5.

In cell B2 the formula remains the same as ............

The final display is then ............

\[
\begin{array}{ccc}
 n & x & f(x) & f'(x) \\
 0 & 1.5 & -0.625 & 7.75 \\
 1 & 1.580645 & 0.042798 & 8.817898 \\
 2 & 1.575792 & 0.000159 & 8.752524 \\
 3 & 1.575773 & 2.21E-09 & 8.75228 \\
 4 & 1.575773 & -8.9E-16 & 8.75228 \\
\end{array}
\]

We cannot be sure that the value 1.575773 is accurate to the sixth decimal place so we must extend the table.

Highlight cells A5 to D5, click Edit on the Command bar and select Copy from the drop-down menu.

Place the cell highlight in cell A6, click Edit and then Paste.

The seventh row of the spreadsheet then fills to produce the display

\[
\begin{array}{ccc}
 n & x & f(x) & f'(x) \\
 0 & 1.5 & -0.625 & 7.75 \\
 1 & 1.580645 & 0.042798 & 8.817898 \\
 2 & 1.575792 & 0.000159 & 8.752524 \\
 3 & 1.575773 & 2.21E-09 & 8.75228 \\
 4 & 1.575773 & -8.9E-16 & 8.75228 \\
\end{array}
\]

And the repetition of the \( x \)-value ensures that the solution \( x = 1.575773 \) is indeed accurate to 6 dp.

Now do one completely on your own.

\[\text{Next frame}\]

**Example 3**

The equation \( 2x^3 - 7x^2 - x + 12 = 0 \) has a root near to \( x = 1.5 \). Use the Newton–Raphson method to find the root to six decimal places.

The spreadsheet solution produces ............
Because
Fill cells A2 to A6 with the numbers 0 to 4
In cell B2 enter the value for \( x_0 \), namely 1.5
In cell C2 enter the formula for \( f(x_0) \), namely \( \frac{1}{4} \times 2\B2^3 - 7\B2^2 - \B2 + 12 \) and copy into cells C3 to C6
In cell D2 enter the formula for \( f'(x_0) \), namely \( 6\B2^2 - 14\B2 - 1 \) and copy into cells D3 to D6
In cell B3 enter the formula for \( x_1 \), namely \( \frac{\B2 - C2}{D2} \) and copy into cells B4 to B6.

The final display is ............

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x )</th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.5</td>
<td>1.5</td>
<td>-8.5</td>
</tr>
<tr>
<td>1</td>
<td>1.676471</td>
<td>0.073275</td>
<td>-7.60727</td>
</tr>
<tr>
<td>2</td>
<td>1.686103</td>
<td>0.000286</td>
<td>-7.54778</td>
</tr>
<tr>
<td>3</td>
<td>1.686141</td>
<td>4.46E-09</td>
<td>-7.54755</td>
</tr>
<tr>
<td>4</td>
<td>1.686141</td>
<td>0</td>
<td>-7.54755</td>
</tr>
</tbody>
</table>

As soon as the number in the second column is repeated then we know that we have arrived at that particular level of accuracy. The required root is therefore \( x = 1.686141 \) to 6 dp.

**First approximations**
The whole process hinges on knowing a ‘starter’ value as first approximation. If we are not given a hint, this information can be found by either
(a) applying the remainder theorem if the function is a polynomial
(b) drawing a sketch graph of the function.

**Example 4**
Find the real root of the equation \( x^3 + 5x^2 - 3x - 4 = 0 \) correct to six significant figures.

Application of the remainder theorem involves substituting \( x = 0, x = \pm 1, x = \pm 2, \) etc. until two adjacent values give a change in sign.

\[
\begin{align*}
f(x) &= x^3 + 5x^2 - 3x - 4 \\
f(0) &= -4; \quad f(1) = -1; \quad f(-1) = 3
\end{align*}
\]

The sign changes from \( f(0) \) to \( f(-1) \). There is thus a root between \( x = 0 \) and \( x = -1 \).
Therefore choose \( x = -0.5 \) as the first approximation and then proceed as before.

Complete the table and obtain the root
\[
x = ............
\]
The final spreadsheet display is

<table>
<thead>
<tr>
<th>n</th>
<th>x</th>
<th>f(x)</th>
<th>f'(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-0.5</td>
<td>1.375</td>
<td>-7.25</td>
</tr>
<tr>
<td>1</td>
<td>-0.689655</td>
<td>0.11907</td>
<td>-8.469679</td>
</tr>
<tr>
<td>2</td>
<td>-0.675597</td>
<td>0.000582</td>
<td>-8.386675</td>
</tr>
<tr>
<td>3</td>
<td>-0.675527</td>
<td>1.43E-08</td>
<td>-8.386262</td>
</tr>
<tr>
<td>4</td>
<td>-0.675527</td>
<td>0</td>
<td>-8.386262</td>
</tr>
</tbody>
</table>

Example 5

Solve the equation $e^x + x - 2 = 0$ giving the root to 6 significant figures.

It is sometimes more convenient to obtain a first approximation to the required root from a sketch graph of the function, or by some other graphical means.

In this case, the equation can be rewritten as $e^x = 2 - x$ and we therefore sketch graphs of $y = e^x$ and $y = 2 - x$.

<table>
<thead>
<tr>
<th>x</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^x$</td>
<td>1.22</td>
<td>1.49</td>
<td>1.82</td>
<td>2.23</td>
<td>2.72</td>
</tr>
<tr>
<td>$2 - x$</td>
<td>1.8</td>
<td>1.6</td>
<td>1.4</td>
<td>1.2</td>
<td>1</td>
</tr>
</tbody>
</table>

It can be seen that the two curves cross over between $x = 0.4$ and $x = 0.6$.

Approximate root $x = 0.4$

$$f(x) = e^x + x - 2 \quad f'(x) = e^x + 1$$

$x = \ldots \ldots \ldots$

Finish it off
The final spreadsheet display is

<table>
<thead>
<tr>
<th>n</th>
<th>x</th>
<th>f(x)</th>
<th>f'(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.4</td>
<td>-0.10818</td>
<td>2.491825</td>
</tr>
<tr>
<td>1</td>
<td>0.443412</td>
<td>0.001426</td>
<td>2.558014</td>
</tr>
<tr>
<td>2</td>
<td>0.442854</td>
<td>2.42E-07</td>
<td>2.557146</td>
</tr>
<tr>
<td>3</td>
<td>0.442854</td>
<td>7.11E-15</td>
<td>2.557146</td>
</tr>
</tbody>
</table>

Note: There are times when the normal application of the Newton–Raphson method fails to converge to the required root. This is particularly so when $f'(x_0)$ is very small, so before we leave this section let us consider this difficulty.

### Modified Newton–Raphson method

If the slope of the curve at $x = x_0$ is small, the value of the second approximation $x = x_1$ may be further from the exact root at A than the first approximation.

If $x = x_0$ is an approximate solution of $f(x) = 0$ and $x = x_0 - h$ is the exact solution then $f(x_0 - h) = 0$. By Taylor’s series

$$f(x_0 - h) = f(x_0) - hf'(x_0) + \frac{h^2}{2!}f''(x_0) - \ldots = 0$$

(a) If we assume that $h$ is small enough to neglect terms of the order $h^2$ and higher then this equation can be written as

$$f(x_0 - h) \approx f(x_0) - hf'(x_0),$$

that is $f(x_0) - hf'(x_0) \approx 0$ and so

$$h \approx \frac{f(x_0)}{f'(x_0)}$$

giving $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ as a better approximation to the solution of $f(x) = 0$.

This is, of course, the relationship we have been using and which may fail when $f'(x)$ is small.

Notice: $h$ is positive unless the sign of $f(x_0)$ is the opposite of the sign of $f'(x_0)$. 

---

**Numerical solutions of equations and interpolation**
(b) If we consider the first three terms then
\[ f(x_0 - h) \approx f(x_0) - hf'(x_0) + \frac{h^2}{2!}f''(x_0) \approx 0, \]
that is
\[ 2f(x_0) - 2hf'(x_0) + h^2f''(x_0) \approx 0 \]
Since \( f'(x_0) \) is small we shall assume that we can neglect it so
\[ h = \pm \sqrt{-\frac{2f(x_0)}{f''(x_0)}} \]
That is \( h = \sqrt{-\frac{2f(x_0)}{f''(x_0)}} \) unless the signs of \( f(x_0) \) and \( f'(x_0) \) are different when it is \( h = -\sqrt{-\frac{2f(x_0)}{f''(x_0)}} \). We use this result only when \( f'(x_0) \) is found to be very small. Having found \( x_1 \) from \( x_0 \) we then revert to the normal relationship \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \) for subsequent iterations.

**Note this**

### Example 6

The equation \( x^3 - 1.3x^2 + 0.4x - 0.03 = 0 \) is known to have a root near \( x = 0.7 \). Determine the root to 6 significant figures.

We start off in the usual way.
\[ f(x) = x^3 - 1.3x^2 + 0.4x - 0.03 \]
\[ f'(x) = 3x^2 - 2.6x + 0.4 \]
and complete the first line of the normal table.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( f(x_n) )</th>
<th>( f'(x_n) )</th>
<th>( h = \frac{f(x_n)}{f'(x_n)} )</th>
<th>( x_{n+1} = x_n - h )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.7</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Complete just the first line of values.

We have

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( f(x_n) )</th>
<th>( f'(x_n) )</th>
<th>( h = \frac{f(x_n)}{f'(x_n)} )</th>
<th>( x_{n+1} = x_n - h )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.7</td>
<td>-0.044</td>
<td>0.05</td>
<td>-0.88</td>
<td>1.58</td>
</tr>
</tbody>
</table>

We notice at once that
(a) The value of \( x_1 \) is well away from the approximate value (0.7) of the root.
(b) The value of \( f'(x_0) \) is small, i.e. 0.05.
To obtain \( x_1 \) we therefore make a fresh start, using the modified relationship \( x_1 = \ldots \ldots \ldots \ldots \ldots \).
We used the modified method to find \( x_1 = x_0 \pm h \), we chose the positive sign since at \( x_0 = 0.7 \), \( f(x_0) \) is negative and the slope \( f'(x_0) \) is positive.

Having established that \( x_1 = 0.9345 \), we now revert to the usual \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \) for the rest of the calculation. Complete the table therefore and obtain the required root.

The final spreadsheet display is

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x )</th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
<th>( f''(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.7</td>
<td>( -0.044 )</td>
<td>( 0.05 )</td>
<td>( 1.6 )</td>
</tr>
<tr>
<td>1</td>
<td>0.934521</td>
<td>0.024625</td>
<td>0.590233</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.892801</td>
<td>0.002544</td>
<td>0.469997</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.887387</td>
<td>4.02E-05</td>
<td>0.45516</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.887298</td>
<td>1.06E-08</td>
<td>0.454919</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.887298</td>
<td>9.16E-16</td>
<td>0.454919</td>
<td></td>
</tr>
</tbody>
</table>

Therefore to six decimal places the required root is \( x = 0.887298 \).

Note that we only used the modified method to find \( x_1 \). After that the normal relationship is used.
And now...

To date our task has been to find a value of $x$ that satisfies an explicit equation $f(x) = 0$. This is quite general because any equation in $x$ can be written in this form. For example, the equation

$$\sin x = x - e^{3x}$$

can always be written as

$$\sin x - x + e^{3x} = 0$$

and then approached by one of the methods that we have discussed so far.

What we want to do now is to work the other way – given a value of $x$, to find the corresponding value of $f(x)$. If $f(x)$ is given explicitly then this is no problem, it is just a matter of substituting the value of $x$ in the formula and working it out. However, many times a function exists but it is not given explicitly, as in the case of a set of readings compiled as a result of an experiment or practical test. We shall consider this problem in the following frames.

Next frame

Interpolation

When a function is defined by a well-understood expression such as

$$f(x) = 4x^3 - 3x^2 + 7$$

or

$$f(x) = 5 \sin(\exp[x])$$

the values of the dependent variable $f(x)$ corresponding to given values of the independent variable $x$ can be found by direct substitution. Sometimes, however, a function is not defined in this way but by a collection of ordered pairs of numbers.

Example 1

A function can be defined by the following set of data:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>3</td>
<td>40</td>
</tr>
<tr>
<td>4</td>
<td>88</td>
</tr>
<tr>
<td>5</td>
<td>164</td>
</tr>
<tr>
<td>6</td>
<td>274</td>
</tr>
</tbody>
</table>

Intermediate values, for example, $x = 2.5$, can be estimated by a process called **interpolation**.

The value of $f(2.5)$ will clearly lie between 14 and 40, the function values for $x = 2$ and $x = 3$.

Purely as an estimate, $f(2.5) = \ldots \ldots..$ What do you suggest?
1 Linear interpolation

If you gave the result as 27, you no doubt agreed that \( x = 2.5 \) is midway between \( x = 2 \) and \( x = 3 \), and that therefore \( f(2.5) \) would be midway between 14 and 40, i.e. 27. This is the simplest form of interpolation, but there is no evidence that there is a linear relationship between \( x \) and \( f(x) \), and the result is therefore suspect.

Of course, we could have estimated the function value at \( x = 2.5 \) by other means, such as

\[ \text{by drawing the graph of } f(x) \text{ against } x \]

2 Graphical interpolation

We could, indeed, plot the graph of \( f(x) \) against \( x \) and, from it, estimate the value of \( f(x) \) at \( x = 2.5 \).

This method is also approximate and can be time consuming.

\[ f(2.5) \approx 26 \]

In what follows we shall look at interpolation using finite differences, which work well and quickly when the values of \( x \) are equally spaced. When the values of \( x \) are not equally spaced we need to resort to the more involved algebraic method called Lagrangian interpolation (which could also be used for equally spaced points).

3 Gregory–Newton interpolation formula using forward finite differences

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_0 )</td>
<td>( f(x_0) )</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>( f(x_1) )</td>
</tr>
</tbody>
</table>

We assume that \( x_0, x_1, \ldots \) are distinct, equally spaced apart, and \( x_0 < x_1 < \ldots \)
For each pair of consecutive function values, \( f(x_0) \) and \( f(x_1) \), in the table, the forward difference \( \Delta f_0 \) is calculated by subtracting \( f(x_0) \) from \( f(x_1) \). This difference is written in a third column of the table, midway between the lines carrying \( f(x_0) \) and \( f(x_1) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( \Delta f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>14</td>
<td>26</td>
</tr>
<tr>
<td>3</td>
<td>40</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>

Complete the table for the data given in Frame 59 which then becomes ............

We now form a fourth column, the forward differences of the values of \( \Delta f \), denoted by \( \Delta^2 f \), and again written midway between the lines of \( \Delta f \). These are the second forward differences of \( f(x) \).

So the table then becomes ............

A further column can now be added in like manner, giving the third differences, denoted by \( \Delta^3 f \), so that we then have ............
Notice that the table has now been completed, for the third differences are constant and all subsequent differences would be zero.

Now we shall see how to use the table. So move on

To find \( f(2.5) \)

We have to find \( f(2.5) \). Therefore denote \( x = 2 \) as \( x_0 \)
\( x = 3 \) as \( x_1 \) \( x = 2.5 \) as \( x_p \)

Let \( h \) = the constant range between successive values of \( x \), i.e. \( h = x_1 - x_0 \)

Express \( (x_p - x_0) \) as a fraction of \( h \), i.e. \( p = \frac{x_p - x_0}{h} \), \( 0 < p < 1 \)

Therefore, in the case above, \( h = 1 \) and \( p = \frac{2.5 - 2.0}{1} = 0.5 \).

All we now use from the table is the set of values underlined by the broken line drawn diagonally from \( f(x_0) \).

So we have
\[
\begin{align*}
p &= \ldots \ldots ; & f_0 &= \ldots \ldots ; & \Delta f_0 &= \ldots \ldots ; \\
\Delta^2 f_0 &= \ldots \ldots ; & \Delta^3 f_0 &= \ldots \ldots
\end{align*}
\]
Now we are ready to deal with the Gregory–Newton forward difference interpolation formula

\[ f_p = f_0 + p\Delta f_0 + \frac{p(p - 1)}{1 \times 2} \Delta^2 f_0 + \frac{p(p - 1)(p - 2)}{1 \times 2 \times 3} \Delta^3 f_0 + \ldots \]

This is sometimes written in operator form

\[ f_p = \left \{ 1 + p\Delta + \frac{p(p - 1)}{2!} \Delta^2 + \frac{p(p - 1)(p - 2)}{3!} \Delta^3 + \ldots \right \} f_0 \]

which you no doubt recognise as the binomial expansion of

\[ f_p = (1 + \Delta)^p f_0\]

Substituting the values in the above example gives

\[ f(2.5) = f_p = \ldots \]

Because

\[ f_p = 14 + 0.5(26) + \frac{0.5(-0.5)}{1 \times 2}(22) + \frac{0.5(-0.5)(-1.5)}{1 \times 2 \times 3}(6) \]

\[ = 14 + 13 - 2.75 + 0.375 \]

\[ = 27.375 - 2.75 = 24.625 \]

Comparing the results of the three methods we have discussed
(a) Linear interpolation \( f(2.5) = 27 \)
(b) Graphical interpolation \( f(2.5) = 26 \)
(c) Gregory–Newton formula \( f(2.5) = 24.625 \) – the true value

**Example 2**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>4</td>
<td>88</td>
</tr>
<tr>
<td>6</td>
<td>274</td>
</tr>
<tr>
<td>8</td>
<td>620</td>
</tr>
<tr>
<td>10</td>
<td>1174</td>
</tr>
</tbody>
</table>

It is required to determine the value of \( f(x) \) at \( x = 5.5 \).

In this case

\[ x_0 = \ldots \quad x_1 = \ldots \]

\[ h = \ldots \quad p = \ldots \]

Because

\[ h = x_1 - x_0 = 6 - 4 = 2 \]

\[ p = \frac{x_p - x_0}{h} = \frac{5.5 - 4}{2} = \frac{1.5}{2} = 0.75 \]

First compile the table of forward differences \( \ldots \)
The Gregory–Newton forward difference interpolation formula is

$$f_p = (1 + \Delta)^p \times f_0$$

i.e. $$f_p = \ldots \ldots$$

$$f_p = \left\{ 1 + p\Delta + \frac{p(p-1)}{2!} \Delta^2 + \frac{p(p-1)(p-2)}{3!} \Delta^3 + \ldots \right\} f_0$$

$$= f_0 + p\Delta f_0 + \frac{p(p-1)}{2!} \Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 f_0 + \ldots$$

So, substituting the relevant values from the table, gives

$$f(5.5) = f_p = \ldots \ldots$$

Because

$$f(5.5) = f_p = 88 + 0.75(186) + \frac{0.75(-0.25)}{1 \times 2} (160)$$

$$+ \frac{0.75(-0.25)(-1.25)}{1 \times 2 \times 3} (48)$$

$$= 88 + 139.5 - 15 + 1.875 = 214.375$$

$$\therefore f(5.5) = 214.4$$

Finally, one more.
**Example 3**

Determine the value of $f(-1)$ from the set of function values.

<table>
<thead>
<tr>
<th>$x$</th>
<th>4</th>
<th>2</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>541</td>
<td>55</td>
<td>1</td>
<td>-53</td>
<td>-155</td>
<td>31</td>
<td>1225</td>
</tr>
</tbody>
</table>

Complete the working and then check with the next frame.

Here is the working; method as before.

\[
x_0 = -2; \quad x_1 = 0; \quad x_p = -1; \quad \therefore h = 2; \quad p = \frac{1}{2}
\]

\[
f_p = f_0 + p\Delta f_0 + \frac{p(p - 1)}{1 \times 2} \Delta^2 f_0 + \frac{p(p - 1)(p - 2)}{1 \times 2 \times 3} \Delta^3 f_0
\]

\[
+ \frac{p(p - 1)(p - 2)(p - 3)}{1 \times 2 \times 3 \times 4} \Delta^4 f_0
\]

\[
= 55 + \frac{1}{2}(-54) + \frac{1}{2}(-\frac{1}{2})(-30) + \frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-48)
\]

\[
+ \frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{3}{2})(384)
\]

\[
= 55 - 27 + 0 - 3 - 15 = 10
\]

\[
\therefore f_p = f(-1) = 10
\]

This table of data does have its restrictions. For example, if we had wanted to find $f(2.5)$ from the table we would have run out of data because there is no $\Delta^4 f$ entry available. In such a case we can resort to a zig-zag path through the table using **central differences**.
Central differences

The central difference operator $\delta$ is defined by its action on the expression $f(x)$ as

$$\delta f(x) = f(x + h/2) - f(x - h/2)$$

and using this operator the interpolated value of $f(x)$ near to the given value of $f_0$ is defined by the Gauss forward formula as

$$f_p = f_0 + p \delta f_0 + \frac{p(p - 1)}{2!} \delta^2 f_0 + \frac{(p + 1)p(p - 1)}{3!} \delta^3 f_0 + \ldots$$

or by the Gauss backward formula as

$$f_p = f_0 + p \delta f_{0-\frac{1}{2}} + \frac{(p + 1)p}{2!} \delta^2 f_0 + \frac{(p + 1)p(p - 1)}{3!} \delta^3 f_{0-\frac{1}{2}} + \frac{(p + 2)(p + 1)p(p - 1)}{4!} \delta^4 f_0 + \ldots$$

There are no tabulated values at the half-interval values $x_0 + h/2$ and $x_0 - h/2$ and so these are taken to be the differences evaluated at mid-interval as given in the forward difference table. This means that the tables for the Gregory–Newton forward differences and the central differences are identical (apart, that is, from the column headings); the method of tracing through the table, however, is different. For example, to find $f(2.5)$ for the example given in Frame 59

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$\delta f(x)$</th>
<th>$\delta^2 f(x)$</th>
<th>$\delta^3 f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>14</td>
<td>10</td>
<td>16</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>40</td>
<td>26</td>
<td>22</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>88</td>
<td>48</td>
<td>28</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>164</td>
<td>76</td>
<td>34</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>274</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Here $x_0 = 2$, $f_0 = 14$, $\delta f_{0+\frac{1}{2}} = 26$, $\delta^2 f_0 = 16$, $\delta^3 f_{0+\frac{1}{2}} = 6$, $\delta^4 f_0 = 0$ and $p = 0.5$. Thus

$$f_p = 14 + (0.5)26 + \frac{(0.5)(-0.5)}{2} 16 + \frac{(0.5)(-0.5)(-1.5)}{6} 6$$

$$= 14 + 13 - 2 - 0.375 = 24.625$$

which agrees with the value found using the Gregory–Newton forward difference formula.
Try one for yourself. The given tabulated values are

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>f(x)</th>
<th>δf(x)</th>
<th>δ²f(x)</th>
<th>δ³f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-5</td>
<td>3</td>
<td>6</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
<td>9</td>
<td>18</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>27</td>
<td>30</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>34</td>
<td>57</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>91</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Using the Gauss forward difference formula, the interpolated value of

\[ f(2.2) = \ldots \ldots \ldots \]

Because

Using \[ f_p = f_0 + p \delta f_{0+\frac{1}{2}} + \frac{p(p - 1)}{2!} \delta^2 f_0 + \frac{p(p - 1)(p + 1)}{3!} \delta^3 f_{0+\frac{1}{2}} + \ldots \]

and following the solid line through the table where

\[ x_0 = 2, \quad f_0 = 7, \quad \delta f_{0+\frac{1}{2}} = 27, \quad \delta^2 f_0 = 18, \quad \delta^3 f_{0+\frac{1}{2}} = 12 \quad \text{and} \quad p = 0.2, \]

then

\[ f_p = 7 + (0.2)27 + \frac{(0.2)(-0.8)}{2}18 + \frac{(0.2)(-0.8)(1.2)}{6}12 \]

\[ = 7 + 5.4 - 1.44 - 0.384 \]

\[ = 10.576 \]

Using the Gauss backward difference formula (following the broken line)

\[ f_p = f_0 + p \delta f_{0-\frac{1}{2}} + \frac{p(p + 1)}{2!} \delta^2 f_0 + \frac{p(p - 1)(p + 1)}{3!} \delta^3 f_{0-\frac{1}{2}} + \ldots \]

where here \( \delta f_{0-\frac{1}{2}} = 9 \) and \( \delta^3 f_{0-\frac{1}{2}} = 12 \) and so

\[ f_p = 7 + (0.2)9 + \frac{(0.2)(1.2)}{2}18 + \frac{(0.2)(1.2)(-0.8)}{6}12 \]

\[ = 7 + 1.8 + 2.16 - 0.384 = 10.576 \]

as found with the Gauss forward difference formula.
Gregory–Newton backward differences

We have seen that the Gregory–Newton forward difference procedure loses terms if the interpolation is for points sufficiently forward in the table. We have also seen how this difficulty can be avoided by using central differences. However, even with central differences we can run out of data before completing a full traverse of the table. In such a situation we resort to the Gregory–Newton backward difference formula

\[ f_p = f_0 + p\Delta f_{-1} + \frac{p(p + 1)}{2!} \Delta^2 f_{-2} + \frac{p(p + 1)(p + 2)}{3!} \Delta^3 f_{-3} + \ldots \]

As an example, consider the table of Frame 74.

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
<th>(\Delta f)</th>
<th>(\Delta^2 f)</th>
<th>(\Delta^3 f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>14</td>
<td>26</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>40</td>
<td>48</td>
<td>22</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>88</td>
<td>76</td>
<td>28</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>164</td>
<td></td>
<td>34</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>274</td>
<td></td>
<td>110</td>
<td></td>
</tr>
</tbody>
</table>

Using this table we can calculate \(f(5.5)\) by tracing back through the table (see broken line) as

\[
f(5.5) = f_0 + (0.5)\Delta f_{-1} + \frac{(0.5)(1.5)}{2} \Delta^2 f_{-2} + \frac{(0.5)(1.5)(2.5)}{6} \Delta^3 f_{-3}
= 164 + (0.5)76 + \frac{(0.5)(1.5)28}{2} + \frac{(0.5)(1.5)(2.5)6}{6}
= 214.375
\]

In each of the examples that we have looked at so far the tabular display of differences eventually results in a column of zeros and this determines the number of terms in an interpolation calculation. The zeros have arisen because all the examples have been derived from polynomials. The following example deals with a tabular display of differences which does not result in a column of zeros. In this case the number of terms used in the interpolation calculation determines confidence in the accuracy of the result.
Example

Use the Gregory–Newton forward difference method to find \( f(0.15) \) to 4 decimal places from the following finite difference table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( \Delta f )</th>
<th>( \Delta^2 f )</th>
<th>( \Delta^3 f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.000000</td>
<td>0.099833</td>
<td>-0.000998</td>
<td>-0.000988</td>
</tr>
<tr>
<td>0.1</td>
<td>0.099833</td>
<td>0.098836</td>
<td>-0.001985</td>
<td>-0.000968</td>
</tr>
<tr>
<td>0.2</td>
<td>0.198669</td>
<td>0.096851</td>
<td>-0.002953</td>
<td>-0.000938</td>
</tr>
<tr>
<td>0.3</td>
<td>0.295520</td>
<td>0.093898</td>
<td>-0.003891</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.389418</td>
<td>0.090007</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.479426</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Here \( x_0 = 0.1 \), \( x_1 = 0.2 \), \( x_p = 0.15 \) and therefore \( p = 0.5 \), and

\[
f_p = f_0 + p \Delta f_0 + \frac{p(p - 1)}{2!} \Delta^2 f_0 + \frac{p(p - 1)(p - 2)}{3!} \Delta^3 f_0 + \ldots
\]

\[
= 0.099833 + \frac{1}{2} (0.098836) + \left( \frac{1}{2} \right) \left( -\frac{1}{2} \right) (-0.001985)/2
\]

\[
+ \left( \frac{1}{2} \right) \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) (-0.000969)/6 + \ldots
\]

\[
= 0.099833 + 0.049418 + 0.000248 - 0.000061 + \ldots
\]

\[
= 0.1494 \text{ to 4 dp}
\]

As you can see, the calculation can continue indefinitely and termination is dictated by the number of decimal places required in the final answer.

Lagrange interpolation

If the straight line \( p(x) = a_0 + a_1x \) passes through the two points \( (x_0, f(x_0)) \) and \( (x_1, f(x_1)) \), where \( a_0 \) and \( a_1 \) are constants, then the equation for this line can also be written as

\[
p(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)
\]

For example, the straight line \( p(x) = 3 + 2x \) passes through the two points \( (1, 5) \) and \( (2, 7) \). Substituting the values for the variables in the above equation demonstrates this alternative form for the equation.
\[ p(x) = \frac{x - 2}{1 - 2} + \frac{x - 1}{2 - 1} \cdot 7 = 10 - 5x + 7x - 7 = 3 + 2x \]

So, given the two data points from Frame 59, \((2, 14)\) and \((3, 40)\), using linear interpolation

\[ f(2.5) \approx p(2.5) = \ldots \ldots \]

Because

\[
\begin{align*}
p(x) &= \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) \\
&= \frac{x - 3}{2 - 3} \cdot 14 + \frac{x - 2}{3 - 2} \cdot 40 = 26x - 38
\end{align*}
\]

and so

\[ f(2.5) \approx p(x) = 26(2.5) - 38 = 27 \]

The principle of Lagrange interpolation is that a function \(f(x)\) whose values are given at a collection of points is assumed to be approximately represented by a polynomial \(p(x)\) that passes through each and every point. The polynomial is called the **interpolation polynomial** and it is of degree one less than the number of points given. For two data points the interpolating polynomial is taken to be a linear polynomial, as you have just seen in the last example. For three data points the interpolating polynomial is taken to be a quadratic, for four data points the interpolation polynomial is taken to be a cubic, and so on.

In the same manner as before it can be shown that the quadratic

\[ p(x) = a_0 + a_1x + a_2x^2 \]

that passes through the three points \((x_0, f(x_0)), (x_1, f(x_1))\) and \((x_2, f(x_2))\) can be written as

\[
\begin{align*}
p(x) &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) \\
&\quad + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)
\end{align*}
\]

So let’s try one. Given the collection of values

<table>
<thead>
<tr>
<th>(x)</th>
<th>(f(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>0.405</td>
</tr>
<tr>
<td>2.1</td>
<td>0.742</td>
</tr>
<tr>
<td>3</td>
<td>1.099</td>
</tr>
</tbody>
</table>

by Lagrangian interpolation, \(f(1.8) \approx \ldots \ldots \) to 2 decimal places.
Because

\[ p(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \]

\[ = \frac{(x - 2 \cdot 1)(x - 3)}{(1 \cdot 5 - 2 \cdot 1)(1 \cdot 5 - 3)} 0.405 + \frac{(x - 1 \cdot 5)(x - 3)}{(2 \cdot 1 - 1 \cdot 5)(2 \cdot 1 - 3)} 0.742 \]

\[ + \frac{(x - 1 \cdot 5)(x - 2 \cdot 1)}{(3 - 1 \cdot 5)(3 - 2 \cdot 1)} 1.099 \]

\[ = \frac{(x^2 - 5 \cdot 1x + 6 \cdot 3)}{0.9} 0.405 + \frac{(x^2 - 4 \cdot 5x + 4 \cdot 5)}{(-0.54)} 0.742 \]

\[ + \frac{(x^2 - 3 \cdot 6x + 3 \cdot 15)}{1.35} 1.099 \]

\[ = -0.11x^2 + 0.958x - 0.784 \]

So that

\[ f(1.8) \approx p(1.8) = 0.58 \] to 2 decimal places.

By carefully considering the interpolating polynomials for two and three data points you should be able to see a pattern. Write down what you think the interpolating polynomial should be for four data points:

\[ \ldots \ldots \ldots \]

Use this interpolating polynomial for the data points

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.368</td>
</tr>
<tr>
<td>1.2</td>
<td>0.301</td>
</tr>
<tr>
<td>1.3</td>
<td>0.273</td>
</tr>
<tr>
<td>1.5</td>
<td>0.223</td>
</tr>
</tbody>
</table>

To 2 decimal places, \( f(1.4) \approx \ldots \ldots \ldots \)
Because $p(x)$

\[
\begin{align*}
&= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1) \\
&\quad + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3) \\
&= \frac{(x-1.2)(x-1.3)(x-1.5)}{(1-1.2)(1-1.3)(1-1.5)} \cdot 0.368 + \frac{(x-1)(x-1.3)(x-1.5)}{(1.2-1)(1.2-1.3)(1.2-1.5)} \cdot 0.301 \\
&\quad + \frac{(x-1)(x-1.2)(x-1.5)}{(1.3-1)(1.3-1.2)(1.3-1.5)} \cdot 0.273 + \frac{(x-1)(x-1.2)(x-1.3)}{(1.5-1)(1.5-1.2)(1.5-1.3)} \cdot 0.223 \\
&= \frac{x^3-4x^2+5.31x-2.34}{(-0.03)} \cdot 0.368 + \frac{x^3-3.8x^2+4.75x-1.95}{0.006} \cdot 0.301 \\
&\quad + \frac{x^3-3.7x^2+4.5x-1.8}{(-0.006)} \cdot 0.273 + \frac{x^3-3.5x^2+4.06x-1.56}{0.03} \cdot 0.223 \\
&= -0.167x^3 + 0.767x^2 - 1.415x + 1.183
\end{align*}
\]

So that

\[f(1.4) \approx p(1.4) = 0.25\] to 2 decimal places

The general Lagrange interpolation polynomial for \(n+1\) data points at \(x_0, x_1, \ldots, x_n\) is

\[
p(x) = \frac{(x-x_1)(x-x_2)(\cdots)(x-x_n)}{(x_0-x_1)(x_0-x_2)(\cdots)(x_0-x_n)}f(x_0) \\
+ \frac{(x-x_0)(x-x_2)(\cdots)(x-x_n)}{(x_1-x_0)(x_1-x_2)(\cdots)(x_1-x_n)}f(x_1) + \cdots \\
+ \frac{(x-x_0)(x-x_1)(\cdots)(x-x_n)}{(x_n-x_0)(x_n-x_1)(\cdots)(x_n-x_{n-1})}f(x_n)
\]

This now completes the work of this Programme. What follows is a Revision summary and a Can you? checklist. Read the summary carefully and respond to the questions in the checklist. When you feel sure that you are happy with the content of this Programme, try the Test exercise. Take your time, there is no need to hurry. Finally, a collection of Further problems provides valuable additional practice.
1. The Fundamental Theorem of Algebra can be stated as follows:
   Every polynomial expression \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \) can be written as a product of \( n \) linear factors in the form
   \[ f(x) = a_n (x - r_1)(x - r_2)(\cdots)(x - r_n) \]

2. Relations between the coefficients and the roots of a polynomial equation
   Whenever a polynomial with real coefficients \( a_i \) has a complex root it also has the complex conjugate as another root.
   If \( \alpha, \beta, \gamma, \ldots \) are the roots of the equation
   \[ p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_{n-1} x + p_n = 0 \]
   then, provided \( p_0 \neq 0 \)
   - sum of roots = \(-\frac{p_1}{p_0}\)
   - sum of the product of the roots, taken two at a time = \(\frac{p_2}{p_0}\)
   - sum of the product of the roots, taken three at a time = \(-\frac{p_3}{p_0}\)
   - sum of the product of the roots, taken \( n \) at a time = \((-1)^n \frac{p_n}{p_0}\).

3. Cubic equations
   Reduced form
   Every cubic equation of the form \( x^3 + ax^2 + bx + c = 0 \) can be written in reduced form \( y^3 + py + q = 0 \) by using the transformation \( x = y - \frac{a}{3} \).

   Tartaglia’s solution
   Every cubic equation with real coefficients has at least one real root that may be found analytically using Tartaglia’s method. The real root of \( x^3 + ax + b = 0 \) when \( a > 0 \) is
   \[ x = \left\{ -\frac{b}{2} + \sqrt{\frac{a^3}{27} + \frac{b^2}{4}} \right\}^{1/3} + \left\{ -\frac{b}{2} - \sqrt{\frac{a^3}{27} + \frac{b^2}{4}} \right\}^{1/3} \]
   If \( a < 0 \) it is best to resort to numerical methods.

4. Numerical methods
   Bisection
   The bisection method of finding a solution to the equation \( f(x) = 0 \) consists of
   - Finding a value of \( x \) such that \( f(x) < 0 \), say \( x = a \)
   - Finding a value of \( x \) such that \( f(x) > 0 \), say \( x = b \).

   The solution to the equation \( f(x) = 0 \) must then lie between \( a \) and \( b \). Furthermore, it must lie either in the first half of the interval between \( a \) and \( b \) or in the second half.
5 **Numerical solution of equations by iteration**
The process of finding the numerical solution to the equation
\[ f(x) = 0 \]
by iteration is performed by first finding an approximate solution and then using this approximate solution to find a more accurate solution. This process is repeated until a solution is found to the required level of accuracy.

6 **Using a spreadsheet**
Iteration procedures are more efficiently performed using a spreadsheet.

7 **Newton–Raphson iteration method**
If \( x = x_0 \) is an approximate solution to the equation \( f(x) = 0 \), a better approximation \( x_1 \) is given by
\[ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \text{ and in general } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

8 **Modified Newton–Raphson iteration method**
If, in the Newton–Raphson procedure \( f'(x_0) \) is sufficiently small enough to cause the value of \( x_1 \) to be a worse approximation to the solution than \( x_0 \), then \( x_1 \) is obtained from the relationship
\[ x_1 = x_0 \pm \sqrt{\frac{-2f(x_0)}{f''(x_0)}} \]

Subsequent iterations then use \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \).

9 **Interpolation**

**Graphical**

10 **Gregory–Newton interpolation formulas using forward finite differences**
\[ f_p = f_0 + p \Delta f_0 + \frac{p(p-1)}{2!} \Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 f_0 + \cdots \]

11 **Gauss interpolation formulas using central finite differences**

**Gauss forward formula**
\[ f_p = f_0 + p \delta f_{0+\frac{1}{2}} + \frac{p(p-1)}{2!} \delta^2 f_0 + \frac{(p+1)p(p-1)}{3!} \delta^3 f_{0+\frac{1}{2}} \]
\[ + \frac{(p+1)p(p-1)(p-2)}{4!} \delta^4 f_0 + \cdots \]

**Gauss backward formula**
\[ f_p = f_0 + p \delta f_{0-\frac{1}{2}} + \frac{(p+1)p}{2!} \delta^2 f_0 + \frac{(p+1)p(p-1)}{3!} \delta^3 f_{0-\frac{1}{2}} \]
\[ + \frac{(p+2)(p+1)p(p-1)}{4!} \delta^4 f_0 + \cdots \]
12 **Gregory–Newton interpolation formula using backward finite differences**

\[ f_p = f_0 + p \Delta f_{-1} + \frac{p(p + 1)}{2!} \Delta^2 f_{-2} + \frac{p(p + 1)(p + 2)}{3!} \Delta^3 f_{-3} + \cdots \]

13 **Lagrange interpolation**

If the straight line \( p(x) = a_0 + a_1 x \) passes through the two points \((x_0, f(x_0))\) and \((x_1, f(x_1))\), where \(a_0\) and \(a_1\) are constants, then the interpolation polynomial (straight line) for this line can be written as

\[ p(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) \]

The quadratic interpolating polynomial that passes through the three points \((x_0, f(x_0)), (x_1, f(x_1))\) and \((x_2, f(x_2))\) can be written as

\[
\begin{align*}
p(x) &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) \\
&\quad + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)
\end{align*}
\]

The cubic interpolating polynomial that passes through the four data points \((x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2))\) and \((x_3, f(x_3))\) can be written as

\[
\begin{align*}
p(x) &= \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f(x_0) \\
&\quad + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f(x_1) \\
&\quad + \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} f(x_2) \\
&\quad + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} f(x_3)
\end{align*}
\]

The interpolating polynomial that passes through \(n + 1\) data points is

\[
\begin{align*}
p(x) &= \frac{(x - x_1)(x - x_2)(\cdots)(x - x_n)}{(x_0 - x_1)(x_0 - x_2)(\cdots)(x_0 - x_n)} f(x_0) \\
&\quad + \frac{(x - x_0)(x - x_2)(\cdots)(x - x_n)}{(x_1 - x_0)(x_1 - x_2)(\cdots)(x_1 - x_n)} f(x_1) + \cdots \\
&\quad + \frac{(x - x_0)(x - x_1)(\cdots)(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)(\cdots)(x_n - x_{n-1})} f(x_n)
\end{align*}
\]
Can you?

Checklist 1

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can:

- Appreciate the Fundamental Theorem of Algebra?
  Yes ☐ ☐ ☐ ☐ ☐ No

- Find the two roots of a quadratic equation and recognise that for polynomial equations with real coefficients complex roots exist in complex conjugate pairs?
  Yes ☐ ☐ ☐ ☐ ☐ No

- Use the relationships between the coefficients and the roots of a polynomial equation to find the roots of the polynomial?
  Yes ☐ ☐ ☐ ☐ ☐ No

- Transform a cubic equation to reduced form?
  Yes ☐ ☐ ☐ ☐ ☐ No

- Use Tartaglia's solution to find the real root of a cubic equation?
  Yes ☐ ☐ ☐ ☐ ☐ No

- Find the solution of the equation \( f(x) = 0 \) by the method of bisection?
  Yes ☐ ☐ ☐ ☐ ☐ No

- Solve equations involving a single real variable by iteration and use a spreadsheet for efficiency?
  Yes ☐ ☐ ☐ ☐ ☐ No

- Solve equations using the Newton–Raphson iterative method?
  Yes ☐ ☐ ☐ ☐ ☐ No

- Use the modified Newton–Raphson method to find the first approximation when the derivative is small?
  Yes ☐ ☐ ☐ ☐ ☐ No

- Understand the meaning of interpolation and use simple linear and graphical interpolation?
  Yes ☐ ☐ ☐ ☐ ☐ No

- Use the Gregory–Newton interpolation formula using forward and backward differences for equally spaced domain points?
  Yes ☐ ☐ ☐ ☐ ☐ No

- Use the Gauss interpolation formulas using central differences for equally spaced domain points?
  Yes ☐ ☐ ☐ ☐ ☐ No

- Use Lagrange interpolation when the domain points are not equally spaced?
  Yes ☐ ☐ ☐ ☐ ☐ No
1 Given that $x = -1 + j\sqrt{3}$ is one root of a quadratic equation with real coefficients, find the other root and hence the quadratic equation.

2 Solve the cubic equation $2x^3 - 7x^2 - 42x + 72 = 0$.

3 Write the cubic $3x^3 + 5x^2 + 3x + 5$ in reduced form and use Tartaglia's method to find the real root.

4 Use the method of bisection to find a solution to $x^3 - 5 = 0$ correct to 4 significant figures.

5 Use the Newton–Raphson method to find a positive solution of the following equation, correct to 6 decimal places:
   \[ \cos 3x = x^2 \]

6 Use the modified Newton–Raphson method to find the solution correct to 6 decimal places near to $x = 2$ of the equation
   \[ x^3 - 6x^2 + 13x - 9 = 0 \]

7 Given the table of values

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>19</td>
</tr>
<tr>
<td>3</td>
<td>70</td>
</tr>
<tr>
<td>4</td>
<td>171</td>
</tr>
<tr>
<td>5</td>
<td>340</td>
</tr>
<tr>
<td>6</td>
<td>595</td>
</tr>
</tbody>
</table>

   estimate
   (a) \( f(2.5) \) using the Gregory–Newton forward difference formula
   (b) \( f(3.4) \) using the Gauss central difference formula
   (c) \( f(5.6) \) using the Gregory–Newton backward difference formula.

8 Given the table of values

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>-9</td>
</tr>
<tr>
<td>5</td>
<td>-108</td>
</tr>
</tbody>
</table>

use Lagrangian interpolation to estimate the value of \( f(2.2) \).
Further problems 1

1. Given that \( x = \frac{-1 - j\sqrt{3}}{2} \) and \( x = \frac{-1 + j}{\sqrt{2}} \) are two roots of a quartic equation with real coefficients, find the other two roots and hence the quartic equation.

2. Solve the equation \( x^3 - 5x^2 - 8x + 12 = 0 \), given that the sum of two of the roots is 7.

3. Find the values of the constants \( p \) and \( q \) such that the function \( f(x) = 2x^3 + px^2 + qx + 6 \) may be exactly divisible by \((x - 2)(x + 1)\).

4. If \( f(x) = 4x^4 + px^3 - 23x^2 + qx + 11 \) and when \( f(x) \) is divided by \( 2x^2 + 7x + 3 \) the remainder is \( 3x + 2 \), determine the values of \( p \) and \( q \).

5. If one root of the equation \( x^3 - 2x^2 - 9x + 18 = 0 \) is the negative of another, determine the three roots.

6. Solve the equation \( x^3 - 7x^2 - 21x + 27 = 0 \), given that the roots form a geometric sequence.

7. Form the equation whose roots are those of the equation \( x^3 + x^2 + 9x + 9 = 0 \) each increased by 2.

8. Form the equation whose roots exceed by 3 the roots of the equation \( x^3 - 4x^2 + x + 6 = 0 \).

9. If the equation \( 4x^3 - 4x^2 - 5x + 3 = 0 \) is known to have two roots whose sum is 2, solve the equation.

10. Solve the equation \( x^3 - 10x^2 + 8x + 64 = 0 \), given that the product of two of the roots is the negative of the third.

11. Form the equation whose roots exceed by 2 those of the equation \( 2x^3 - 3x^2 - 11x + 6 = 0 \).

12. If \( \alpha, \beta, \gamma \) are the roots of the equation \( x^3 + px^2 + qx + r = 0 \), prove that \( \alpha^2 + \beta^2 + \gamma^2 = p^2 - 2q \).

13. Using Tartaglia’s solution, find the real root of the equation \( 2x^3 + 4x - 5 = 0 \) giving the result to 4 significant figures.

14. Solve the equation \( x^3 - 6x - 4 = 0 \).

15. Rewrite the equation \( x^3 + 6x^2 + 9x + 4 = 0 \) in reduced form and hence determine the three roots.

16. Show that the equation \( x^3 + 3x^2 - 4x - 6 = 0 \) has a root between \( x = 1 \) and \( x = 2 \), and use the Newton–Raphson iterative method to evaluate this root to 4 significant figures.

17. Find the real root of the equations:
   (a) \( x^3 + 4x + 3 = 0 \)  (b) \( 5x^3 + 2x - 1 = 0 \).
18 Solve the following equations:
   (a) \( x^3 - 5x + 1 = 0 \)  
   (b) \( x^3 + 2x - 3 = 0 \)  
   (c) \( x^3 - 4x + 1 = 0 \).

19 Express the following in reduced form and determine the roots:
   (a) \( x^3 + 6x^2 + 9x + 5 = 0 \)  
   (b) \( 8x^3 + 20x^2 + 6x - 9 = 0 \)  
   (c) \( 4x^3 - 9x^2 + 42x - 10 = 0 \).

20 Use the Newton–Raphson iterative method to solve the following.
   (a) Show that a root of the equation \( x^3 + 3x^2 + 5x + 9 = 0 \) occurs between \( x = -2 \) and \( x = -3 \). Evaluate the root to four significant figures.
   (b) Show graphically that the equation \( e^{2x} = 25x - 10 \) has two real roots and find the larger root correct to four significant figures.
   (c) Verify that the equation \( x - \cos x = 0 \) has a root near to \( x = 0.8 \) and determine the root correct to three significant figures.
   (d) Obtain graphically an approximate root of the equation \( 2 \ln x = 3 - x \). Evaluate the root correct to four significant figures.
   (e) Verify that the equation \( x^4 + 5x - 20 = 0 \) has a root at approximately \( x = 1.8 \). Determine the root correct to five significant figures.
   (f) Show that the equation \( x + 3 \sin x = 2 \) has a root between \( x = 0.4 \) and \( x = 0.6 \). Evaluate the root correct to five significant figures.
   (g) The equation \( 2 \cos x = e^x - 1 \) has a real root between \( x = 0.8 \) and \( x = 0.9 \). Evaluate the root correct to four significant figures.
   (h) The equation \( 20x^3 - 22x^2 + 5x - 1 = 0 \) has a root at approximately \( x = 0.6 \). Determine the value of the root correct to four significant figures.

21 A polynomial function is defined by the following set of function values

<table>
<thead>
<tr>
<th>( x )</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = f(x) )</td>
<td>-7.00</td>
<td>9.00</td>
<td>97.0</td>
<td>305</td>
<td>681</td>
</tr>
</tbody>
</table>

Find
   (a) \( f(4.8) \) using the Gregory–Newton forward difference formula
   (b) \( f(7.2) \) using the Gauss central difference formula
   (c) \( f(8.5) \) using the Gregory–Newton backward difference formula.

22 For the function \( f(x) \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>-10</td>
<td>12</td>
<td>56</td>
<td>128</td>
<td>234</td>
<td>380</td>
<td>572</td>
</tr>
</tbody>
</table>

Find
   (a) \( f(4.5) \) and \( f(6.4) \) using the Gregory–Newton forward difference formula
   (b) \( f(7.1) \) and \( f(8.9) \) using the Gregory–Newton backward difference formula.
For the function defined in the table above, evaluate (a) \( f(2.6) \) and (b) \( f(7.2) \).

A function \( f(x) \) is defined by the following table:

<table>
<thead>
<tr>
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<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>−9</td>
<td>35</td>
<td>231</td>
<td>675</td>
<td>1463</td>
<td>2691</td>
</tr>
</tbody>
</table>

Find
(a) \( f(-3) \) and \( f(1.6) \) using the Gregory–Newton forward difference formula
(b) \( f(0.2) \) and \( f(3.1) \) using the Gauss central difference formula
(c) \( f(4.4) \) and \( f(7) \) using the Gregory–Newton backward difference formula.

Given the table of values

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>−1</td>
<td>−2.71828</td>
</tr>
<tr>
<td>3</td>
<td>−0.04979</td>
</tr>
<tr>
<td>5</td>
<td>−0.00674</td>
</tr>
</tbody>
</table>

use Lagrangian interpolation to find the value of \( f(3.4) \).

Given the table of values

<table>
<thead>
<tr>
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</thead>
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<tr>
<td>6</td>
<td>0.801153</td>
</tr>
<tr>
<td>7.2</td>
<td>−0.82236</td>
</tr>
<tr>
<td>9</td>
<td>−0.73922</td>
</tr>
<tr>
<td>13</td>
<td>0.994808</td>
</tr>
</tbody>
</table>

use Lagrangian interpolation to find the value of \( f(8) \).

Given the table of values

<table>
<thead>
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<tbody>
<tr>
<td>−2</td>
<td>−2.63906</td>
</tr>
<tr>
<td>0</td>
<td>−2.48491</td>
</tr>
<tr>
<td>5</td>
<td>−1.94591</td>
</tr>
<tr>
<td>6</td>
<td>−1.79176</td>
</tr>
</tbody>
</table>

use Lagrangian interpolation to find the values of
(a) \( f(-0.8) \)  (b) \( f(0.8) \)  (c) \( f(5.5) \).
Index

Adjoint matrix 532
Alternating sign test 991
Amplitude spectrum 303
Analytic function 937, 978
Analytic methods 10
Angle between two vectors 772
Anisotropic heat diffusion equation 614
Arbitrary input to a continuous system 229
Arbitrary input to a discrete system 230
Area enclosed by a curve 657
Areas of plane figures bounded by a curve 644
Artificial variable 1036, 1055
Augmented coefficient matrix 526, 539

Backward difference formula 33, 594, 632
Bessel functions 380, 393
Bessel's equation 379, 393
Beta function 745, 766
Bilateral Laplace transform 166
Bilinear transformation 927
Bisection 10, 38
Burger's equation 614

Cartesian coordinates 703, 729, 890
Cauchy–Euler equations 349, 354
Cauchy–Riemann equations 939, 978
Cauchy's theorem 949
Cayley–Hamilton theorem 571, 589
Central difference formula 31, 595, 632
Partial derivatives 632
Change of variables 446, 476
Characteristic determinant 564
Characteristic equation 564
Difference equation 161
Characteristic values 564
Circle of convergence 990, 1010
Coefficients of a polynomial 4, 38
Complementary error function 756, 767
Complex analysis 895
Complex conjugate pair 3
Complex Fourier series 298, 327
Two domains 304
Complex integration 946, 978
Complex mapping 897
Complex spectra 303
Amplitude spectrum 303
Phase spectrum 303
Complex transformation 897, 903
Bilinear transformation 927
Conformal mapping 963
Inversion 918
Magnification 904
Magnification and rotation 906
Magnification, rotation and translation 908
Non-linear transformations 912
Rotation 905
Translation 903
Computational molecules 601, 633
Conformal transformation 963, 979
Critical points 964
Schwarz–Christoffel 967
Conservative vector fields 839, 863
Consistency of a set of equations 526, 558
Consolidation equation 613
Continuous complex spectra 328
Continuous spectra 305
Contour integration 946, 978
Convolution 112, 322, 329
Theorem 117
Convolution sum 221
Convolution theorem 230, 329
Fourier transform 323
Laplace transform 119
Coplanar vectors 779, 812
Cover up rule 87
Partial fractions 67
Crank–Nicolson procedure 624, 634
Critical points 964
Cubic equations 7
Reduced form 8, 38
Tartaglia's solution 8, 38
Curl of a vector field 806, 807, 813
Curvilinear coordinates 719, 870
Orthogonal 874
Transformation in three dimensions 727
Cylindrical coordinates 704, 729, 890

Damped motion 144
Derivative boundary conditions 608, 633
Derivative of Laplace transform 55
Derivative of unit step function 139
Derivative of Z transform 175, 187
Diagonalisation of a matrix 577, 589
Difference equations 155, 158, 185, 230
Characteristic equation 161
Inhomogeneous solution 160
Order 158
Particular solution 160, 163
Solution by inspection 160
Solving by Z transform 180
Transfer function 224
Differential equations 228
Cauchy–Euler 349
Frobenius’ method 358
Linear systems 201
Numerical solutions 398
Power series solutions 334
Transfer function 216
Unit impulse 140
Unit step function 108
Differentials 650, 686
Differentiation 329
Complex function 936, 978
Fourier transform 320
Sums and products 789
Vectors 784, 812
Diffusivity constant 619
Dimensional analysis 620, 631
Dirac delta function 134, 151, 314, 328
Graph 135
Direction cosines 772
Directional derivatives 798
Dirichlet conditions 250, 261
Discontinuity 259, 261
Discrete complex spectra 327
Discrete linear system 200
Arbitrary input 220
Exponential response 222
Shift-invariance 200
Transfer function 223
Discrete unit impulse 169, 219
Shifted 219
Discrete unit impulse response 220

Discrete unit step function 168
Divergence of a vector field 805, 807, 813
Divergence theorem 844, 864, 1067
Double integrals 643, 692
Eigenfunction 388
Eigenvalue 388, 564, 589
Eigenvectors 564, 589
Element of volume 730
Cartesian coordinates 707, 730
Cylindrical coordinates 707, 730
Spherical coordinates 707, 730
Elementary operations 557
Elliptic equations 612, 633
Elliptic functions 758, 767
Alternative forms 763, 767
Complete 759, 767
Standard forms 759, 767
Entire function 937
Equations
Interpolation 1, 24
Linear ordinary differential equations 47
Newton–Raphson solution 15
Numerical solution 1
Solution by bisection 10
Solution by iteration 12
Tartaglia’s solution 8
Equivalent matrix 522, 535
Error function (graph) 756, 755, 767
Essential singularity 994
Euler–Cauchy method 416, 417, 434
Euler’s method 401, 434
Errors 410
Graphical interpretation 414
Euler’s second order method 425, 435
Even functions 272, 292
Fourier transform 310
Even harmonics 293
Exact differential 653, 687
Three dimensions 678, 687
Exponential response 212, 229, 230
Discrete linear system 222

Feasible region 1018
Feasible solution 1055
Final value theorem 174, 187
Finite differences 25
Backward 594
Central 595
Forward 25, 594
Index

First order differential equations 72
  Euler–Cauchy method 416
  Euler's method 401
  Laplace transforms 72
  Predictor–corrector methods 432
  Runge–Kutta method 422
First order shift theorem 54, 86, 171, 186
Force harmonic oscillator 146
Forward difference formula 594, 632
Forward finite differences 25
Fourier coefficients 261, 269
Fourier cosine series 293
Fourier cosine transforms 325, 330
Fourier series 236, 261
  Complex exponentials 298
  Complex spectra 303
  Defined 247
  Dirichlet conditions 250
  Even functions 272
  Even harmonics only 286
  Half-range series 282, 289
  Harmonics 238
  Odd functions 272
  Odd harmonics only 286
  Significance of constant term 288
  Sum at a discontinuity 259
Fourier sine series 292
Fourier sine transforms 325, 330
Fourier transform 297
  Amplitude spectrum 308
  Convolution theorem 323
  Defined 307
  Differentiation 320
  Even functions 310
  Frequency shifting 318
  Linearity 317
  Odd functions 310
  Phase spectrum 309
  Symmetry 319
  Table of transforms 327
  Time scaling 318
  Time shifting 318
  Triangle function 316
Fourier's integral theorem 307, 328
Frequency shifting 329
  Fourier transform 318
Frobenius' method 358, 376
  Indicial equation 360
Functions of a complex variable 896
Functions with arbitrary period 268, 292
  Fundamental theorem of algebra 2, 38
Gamma function 380, 736, 766
  Duplication formula 754
  Graph 741
Gauss backward difference formula 31, 39
Gauss forward difference formula 31, 39
Gauss' theorem 844, 864, 1067
Gaussian curve 757
Gaussian elimination method 539
General curvilinear coordinates 882, 891
  Curl 885, 891
  Div 885, 891
  Element of arc ds 884
  Element of volume dV 885
  Grad 885, 891
Generating function 394
  Legendre polynomials 386
Gibbs’ phenomenon 258
Grad, Div, Curl 807
  Cartesian coordinates 889
  Cylindrical coordinates 889, 892
  Spherical coordinates 890, 892
  Sums and products 803
Gradient of a scalar field 795, 813
Graphical interpolation 25
Green’s theorem 679, 687, 859, 864, 1063
Gregory–Newton 25, 33
  Backward differences 33, 40
  Forward differences 28, 39
  Grid values 598, 632
Half-range series 282, 293
  Arbitrary period 289
Harmonic functions 941
Harmonic oscillators 142, 152
  Damped motion 144
  Forced 152
  Forced with damping 146
  Resonance 149
Harmonics 238, 261
  Effect of 257
Heat conduction equation 496, 514, 613
  Solution 497
Heaviside unit step function 93, 118, 321, 328
Helmholtz equation 614
Hyperbolic equations 613, 633
  Imaginary part 49
Implicit functions 445, 476
Impulse response 208
<table>
<thead>
<tr>
<th>Indicial equation</th>
<th>360</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inequalities</td>
<td>1055</td>
</tr>
<tr>
<td>Initial value theorem</td>
<td>175, 187</td>
</tr>
<tr>
<td>Input-response relationships</td>
<td>194</td>
</tr>
<tr>
<td>Integral functions</td>
<td>735</td>
</tr>
<tr>
<td>Integral of Laplace transform</td>
<td>57</td>
</tr>
<tr>
<td>Integrals of periodic functions</td>
<td>243</td>
</tr>
<tr>
<td>Integration of exact differentials</td>
<td>655</td>
</tr>
<tr>
<td>Integration of vector functions</td>
<td>792, 813</td>
</tr>
<tr>
<td>Interpolation</td>
<td>1, 24</td>
</tr>
<tr>
<td>Graphical</td>
<td>25</td>
</tr>
<tr>
<td>Gregory–Newton</td>
<td>25</td>
</tr>
<tr>
<td>Lagrange</td>
<td>25, 34, 40</td>
</tr>
<tr>
<td>Linear</td>
<td>25</td>
</tr>
<tr>
<td>Interpolation polynomial</td>
<td>35</td>
</tr>
<tr>
<td>Invariant linear systems</td>
<td>192</td>
</tr>
<tr>
<td>Inverse functions</td>
<td>450, 477</td>
</tr>
<tr>
<td>Inverse Laplace transforms</td>
<td>60</td>
</tr>
<tr>
<td>Periodic functions</td>
<td>130, 151</td>
</tr>
<tr>
<td>Table of transforms</td>
<td>68, 88</td>
</tr>
<tr>
<td>Inverse matrix</td>
<td>531</td>
</tr>
<tr>
<td>Inverse Z transformations</td>
<td>177</td>
</tr>
<tr>
<td>Inversion transformation</td>
<td>918</td>
</tr>
<tr>
<td>Iteration</td>
<td>12, 39</td>
</tr>
<tr>
<td>Newton–Raphson</td>
<td>15, 39</td>
</tr>
<tr>
<td>Jacobian</td>
<td>453</td>
</tr>
<tr>
<td>Kronecker delta</td>
<td>997</td>
</tr>
<tr>
<td>Lagrange interpolation</td>
<td>25, 34, 40</td>
</tr>
<tr>
<td>Lagrange undetermined multipliers</td>
<td>470, 477</td>
</tr>
<tr>
<td>Function of three variables</td>
<td>473</td>
</tr>
<tr>
<td>Laplace transform</td>
<td>46, 86</td>
</tr>
<tr>
<td>Bilateral</td>
<td>166</td>
</tr>
<tr>
<td>Convolution</td>
<td>117</td>
</tr>
<tr>
<td>Derivative of transform</td>
<td>55, 87</td>
</tr>
<tr>
<td>Derivatives</td>
<td>70, 88</td>
</tr>
<tr>
<td>First shift theorem</td>
<td>54, 86</td>
</tr>
<tr>
<td>Integral of transform</td>
<td>57, 87</td>
</tr>
<tr>
<td>Inverse transforms</td>
<td>60</td>
</tr>
<tr>
<td>Linearity</td>
<td>86</td>
</tr>
<tr>
<td>Multiplication of expression by a constant</td>
<td>53</td>
</tr>
<tr>
<td>Periodic functions</td>
<td>124, 151</td>
</tr>
<tr>
<td>Product</td>
<td>117</td>
</tr>
<tr>
<td>Second shift theorem</td>
<td>98</td>
</tr>
<tr>
<td>Sum or difference of expressions</td>
<td>53</td>
</tr>
<tr>
<td>Table of transforms</td>
<td>59, 86</td>
</tr>
<tr>
<td>Unit impulse</td>
<td>136, 152</td>
</tr>
<tr>
<td>Unit step function</td>
<td>97</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Laplace's equation</th>
<th>502, 514, 612, 941</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plane polars</td>
<td>507, 515</td>
</tr>
<tr>
<td>Separating the variables</td>
<td>508</td>
</tr>
<tr>
<td>Solution</td>
<td>502</td>
</tr>
<tr>
<td>Latent roots</td>
<td>564</td>
</tr>
<tr>
<td>Laurent's series</td>
<td>993, 1011</td>
</tr>
<tr>
<td>Legendre polynomials</td>
<td>386, 394</td>
</tr>
<tr>
<td>Generating function</td>
<td>386</td>
</tr>
<tr>
<td>Legendre's equation</td>
<td>385, 394</td>
</tr>
<tr>
<td>Leibnitz theorem</td>
<td>338, 353</td>
</tr>
<tr>
<td>Leibnitz–Maclaurin method</td>
<td>342, 353</td>
</tr>
<tr>
<td>Line integrals</td>
<td>660, 686, 819, 862</td>
</tr>
<tr>
<td>Alternative form</td>
<td>661</td>
</tr>
<tr>
<td>Arc length</td>
<td>687, 671</td>
</tr>
<tr>
<td>Around a closed curve</td>
<td>667</td>
</tr>
<tr>
<td>Closed curve</td>
<td>686</td>
</tr>
<tr>
<td>Complex plane</td>
<td>946, 978</td>
</tr>
<tr>
<td>Parametric equations</td>
<td>672</td>
</tr>
<tr>
<td>Path dependent</td>
<td>673, 687</td>
</tr>
<tr>
<td>Properties</td>
<td>664, 686</td>
</tr>
<tr>
<td>Scalar field</td>
<td>819</td>
</tr>
<tr>
<td>Vector field</td>
<td>822</td>
</tr>
<tr>
<td>Linear inequalities</td>
<td>1016</td>
</tr>
<tr>
<td>Linear interpolation</td>
<td>25</td>
</tr>
<tr>
<td>Linear programming</td>
<td>1015, 1055</td>
</tr>
<tr>
<td>Constraints</td>
<td>1015</td>
</tr>
<tr>
<td>Feasible region</td>
<td>1018</td>
</tr>
<tr>
<td>Minimisation</td>
<td>1047</td>
</tr>
<tr>
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<td>1015</td>
</tr>
<tr>
<td>Simplex method</td>
<td>1022</td>
</tr>
<tr>
<td>Linear recurrence relation</td>
<td>158</td>
</tr>
<tr>
<td>Linear system</td>
<td>192, 195, 228</td>
</tr>
<tr>
<td>Arbitrary input</td>
<td>208</td>
</tr>
<tr>
<td>Differential equations</td>
<td>201</td>
</tr>
<tr>
<td>Exponential response</td>
<td>212</td>
</tr>
<tr>
<td>Time-invariance</td>
<td>198</td>
</tr>
<tr>
<td>Transfer function</td>
<td>213</td>
</tr>
<tr>
<td>Impulse response</td>
<td>208</td>
</tr>
<tr>
<td>Zero-input solution</td>
<td>203</td>
</tr>
<tr>
<td>Zero-state solution</td>
<td>202</td>
</tr>
<tr>
<td>Linear transformation</td>
<td>931</td>
</tr>
<tr>
<td>Maclaurin series</td>
<td>984, 1010</td>
</tr>
<tr>
<td>Magnification</td>
<td>931</td>
</tr>
<tr>
<td>Mapping of a region</td>
<td>931, 917</td>
</tr>
<tr>
<td>Mapping of a straight line</td>
<td>899</td>
</tr>
<tr>
<td>Matrix</td>
<td>519</td>
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<tr>
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<td>532</td>
</tr>
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</tr>
<tr>
<td>Characteristic determinant</td>
<td>564</td>
</tr>
<tr>
<td>Characteristic equation</td>
<td>564</td>
</tr>
<tr>
<td>Characteristic values</td>
<td>564</td>
</tr>
</tbody>
</table>
Diagonalisation 577
Elementary row operations 522
Equivalent 522, 535
Inverse 531
Latent roots 564
Modal 577
Non-singular 520
Rank 521
Singular 520
Spectral 577
Upper triangular 524
Matrix transformation 553, 559
Maximum values 459
Minimisation 1055
Minimum values 459
Modal matrix 577, 589
Modified Newton–Raphson method 21, 39
Multiple integrals 642
  Change of variables 717, 731
  Double integrals 643
  Triple integrals 643
Newton–Raphson 15, 39
  First approximations 19
  Modified method 21, 39
Non-linear transformations 912, 931
   Bilinear transformation 927
   Inversion 918
Non-singular matrix 520
Numerical methods 1, 10
  Bisection 10, 38
  Computational molecules 601
  Derivative boundary conditions 608
  Functions of two real variables 597
  Grid values 598
  Iteration 12, 39
  Second partial derivatives 615
  Time-dependent solutions 619
Objective function 1015
Odd functions 272, 292
  Fourier transform 310
Odd harmonics 293
Open polygon 972
Optimal solution 1055
Optimization 1055
Optimization and linear programming 1014
Orthogonal curvilinear coordinates 874, 890
  Cylindrical coordinates 876
  Spherical polar coordinates 878
Orthogonal functions 247
Orthogonality 390, 395
Parabolic equations 613, 633
Parametric equations 672
Partial differential equations 482
  Initial and boundary conditions 485
  Numerical solutions 593
  Solution by integration 484
Partial differentiation 439
  Change of variables 446
  Implicit functions 445
  inverse functions 450
  Jacobian 453
  Rate of change 444
  Vectors 792
Partial fractions cover up rule 67, 87
Partial fractions rules 61, 87
Period 237
Periodic functions 237, 261
  Analytical description 239
  Integrals 243
  Laplace transform 124
  Non-sinusoidal 239
  Phase spectrum 303
Poisson’s equation 612
Polar coordinates 863
  Cylindrical 863
  Plane 863
  Spherical 863
Pole 989, 1010
Polynomial 2
  Interpolation 35
  Roots 2
  Sum of products of roots 5
Power series solutions 378
  Differential equations 334
  Frobenius’ method 358
  Leibnitz–Maclaren method 342
Predictor–corrector methods 432, 435
Products of odd and even functions 275
Radius of convergence 988, 1010
Rank of a matrix 521
Rates of change 444, 476
Ratio test 984, 1010
Real part 49
Recurrence relation 343
Recursive process 157
Reduction formula 50, 746
Regions enclosed by closed curves 666
Regular function 937, 978
Removable singularity 990, 1010
Residues 997, 1011
  Calculating 999, 1011
  Integrals of real functions 1001
  Real integrals 1011
Resonance 149, 152
Responses of a continuous system 208
Responses of a discrete system 219
Rodrigue’s formula 386, 394
Roots of a polynomial 2
  Coefficients 4
  Complex 3
Rotation 931
Rules of partial fractions 61
Runge–Kutta method 422, 427, 435
  Errors 423
  Runge–Kutta second order method 435
Saddle point 465
Sampling 183
  Z transforms 187
 Scalar 772
 Scalar field 795, 819, 831
 Scalar product 772, 812
 Scalar triple product 777, 812
  Properties 778
Scale factors 879
  Cylindrical coordinates 880
  Rectangular coordinates 880
  Spherical coordinates 880
Schwarz–Christoffel transformation 967, 979
Second order differential equations 74
  Euler’s method 425
  Laplace transforms 74
  Runge–Kutta method 427
Second shift theorem 98, 119, 173, 186
Separating the variables 487, 515
Sequences 156, 185
  Shifting to the left 171
  Shifting to the right 173
Series of Legendre polynomials 392
Shifted discrete unit impulse 219
Shift-invariance 200, 228
Simplex 1022, 1055
  Artificial variables 1036
  Change of variables 1026
  Computation 1024
  Key column 1024
  Key row 1024
  Pivot 1024
  Slack variables 1022
Simplex tableau 1022
Simultaneous differential equations 81
  Laplace transforms 81, 89
  Sinc function 303
Singular matrix 520
Singularity 937, 978, 989
  Essential 994
  Poles 989, 1010
  Removable singularity 990, 1010
  Slack variable 1022, 1055
Small increments 442, 476
Solution of sets of equations 526
  Gaussian elimination method 535, 589
  Inverse method 531, 558
  Row transformation method 535, 558
  Triangular decomposition method 542, 558
  Using a spreadsheet 548, 558
Solving difference equations 180, 186
  Z transforms 187
Space coordinate systems 703
Specific heat 619
Spectral matrix 577, 589
Spherical coordinates 705, 729, 891
Spreadsheet 12, 406, 548
  absolute addresses 14
  Command bar 12
  MINVERSE(array) 548
  MMULT(array1, array2) 548
  Relative addresses 14
  Tabular display 16
Stationary values 458, 477
  Lagrange undetermined multipliers 470
  Maximum and minimum 459
  Saddle point 465
Stokes’ theorem 850, 864, 1069
Sturm–Liouville systems 388, 394
Surface in space 729
Surface integrals 697, 729, 830, 863
  Scalar field 831
  Vector field 834
Symmetry 329
  Fourier transform 319
Systems 193, 228
  Response 193, 194
  Systems of differential equations 563, 589, 590
  First order 572
  Second order 582
<table>
<thead>
<tr>
<th>Term</th>
<th>Page(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table of Fourier transforms</td>
<td>327</td>
</tr>
<tr>
<td>Table of inverse Laplace transforms</td>
<td>68, 88</td>
</tr>
<tr>
<td>Table of Laplace transforms</td>
<td>59</td>
</tr>
<tr>
<td>Table of Z transforms</td>
<td>170, 186</td>
</tr>
<tr>
<td>Tartaglia’s solution</td>
<td>8</td>
</tr>
<tr>
<td>Taylor’s series</td>
<td>399, 434, 440, 991, 1011</td>
</tr>
<tr>
<td>Telegraph equation</td>
<td>614</td>
</tr>
<tr>
<td>Thermal conductivity</td>
<td>619</td>
</tr>
<tr>
<td>Time-dependent equations</td>
<td>634</td>
</tr>
<tr>
<td>Time scaling</td>
<td>329</td>
</tr>
<tr>
<td>Time shifting</td>
<td>329</td>
</tr>
<tr>
<td>Fourier transform</td>
<td>318</td>
</tr>
<tr>
<td>Time invariance</td>
<td>198, 207, 228</td>
</tr>
<tr>
<td>Top-hat function</td>
<td>312, 328</td>
</tr>
<tr>
<td>Transfer function</td>
<td>213, 214, 230</td>
</tr>
<tr>
<td>Difference equations</td>
<td>224</td>
</tr>
<tr>
<td>Differential equations</td>
<td>216</td>
</tr>
<tr>
<td>Discrete linear system</td>
<td>223</td>
</tr>
<tr>
<td>Transformation equation</td>
<td>883, 931</td>
</tr>
<tr>
<td>Transforms of derivatives</td>
<td>70</td>
</tr>
<tr>
<td>Translation</td>
<td>931</td>
</tr>
<tr>
<td>Triangle function</td>
<td>316, 328</td>
</tr>
<tr>
<td>Triangular decomposition method</td>
<td>542</td>
</tr>
<tr>
<td>Triple integrals</td>
<td>643</td>
</tr>
<tr>
<td>Triple products</td>
<td>777</td>
</tr>
<tr>
<td>Two domains</td>
<td>304</td>
</tr>
<tr>
<td>Uniqueness of solutions</td>
<td>527, 558</td>
</tr>
<tr>
<td>Unit impulse</td>
<td>134, 151</td>
</tr>
<tr>
<td>Differential equations</td>
<td>140</td>
</tr>
<tr>
<td>Discrete</td>
<td>169</td>
</tr>
<tr>
<td>Graph</td>
<td>135</td>
</tr>
<tr>
<td>Integration</td>
<td>152</td>
</tr>
<tr>
<td>Laplace transform</td>
<td>136, 152</td>
</tr>
<tr>
<td>Unit normal vectors</td>
<td>801, 853</td>
</tr>
<tr>
<td>Unit sample</td>
<td>220</td>
</tr>
<tr>
<td>Unit step function</td>
<td>93, 321, 328</td>
</tr>
<tr>
<td>Derivative</td>
<td>139</td>
</tr>
<tr>
<td>Differential equations</td>
<td>108</td>
</tr>
<tr>
<td>Discrete</td>
<td>168</td>
</tr>
<tr>
<td>Effect of</td>
<td>94, 119</td>
</tr>
<tr>
<td>Laplace transform</td>
<td>97, 119</td>
</tr>
<tr>
<td>Unit tangent vector</td>
<td>789, 812</td>
</tr>
<tr>
<td>Unit vectors</td>
<td>812</td>
</tr>
<tr>
<td>Upper triangular matrix</td>
<td>524</td>
</tr>
<tr>
<td>Vector analysis</td>
<td>771</td>
</tr>
<tr>
<td>Vector field</td>
<td>795, 822, 834</td>
</tr>
<tr>
<td>Conservative</td>
<td>839</td>
</tr>
<tr>
<td>Vector product</td>
<td>772, 812</td>
</tr>
<tr>
<td>Vector triple product</td>
<td>781, 812, 1066</td>
</tr>
<tr>
<td>Volume integrals</td>
<td>708, 730, 826, 863</td>
</tr>
<tr>
<td>Volumes of solids</td>
<td>645</td>
</tr>
<tr>
<td>Wave equation</td>
<td>486, 514, 613</td>
</tr>
<tr>
<td>Solution</td>
<td>487</td>
</tr>
<tr>
<td>Z transform</td>
<td>155, 166, 186</td>
</tr>
<tr>
<td>Derivative</td>
<td>175</td>
</tr>
<tr>
<td>Final value theorem</td>
<td>174</td>
</tr>
<tr>
<td>First shift theorem</td>
<td>171</td>
</tr>
<tr>
<td>Initial value theorem</td>
<td>175</td>
</tr>
<tr>
<td>Inverse transforms</td>
<td>177</td>
</tr>
<tr>
<td>Linearity</td>
<td>171</td>
</tr>
<tr>
<td>Properties</td>
<td>171</td>
</tr>
<tr>
<td>Sampling</td>
<td>183</td>
</tr>
<tr>
<td>Second shift theorem</td>
<td>173</td>
</tr>
<tr>
<td>Table of transforms</td>
<td>170</td>
</tr>
<tr>
<td>Translation</td>
<td>174</td>
</tr>
<tr>
<td>Zero-input response</td>
<td>195, 201, 205, 229</td>
</tr>
<tr>
<td>Zero-state response</td>
<td>201</td>
</tr>
</tbody>
</table>